ČESKÉ VYSOKÉ UČENÍ TECHNICKÉ V PRAZE
Fakulta jaderná a fyzikálně inženýrská

DIPLOMOVÁ PRÁCE

2003 Petr Ambrož

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DIPLOMOVÁ PRÁCE

Číselné soustavy spojené s Pisotovými čísly

(Pisot numbers and numeration systems)

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Prohlašuji na tomto místě, že jsem předloženou práci vypracoval samostatně a že jsem uvedl veškerou použitou literaturu.

V Praze dne 12. května 2003 ....
Petr Ambrož

Poděkování

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Contents

List of used symbols

Common symbols for set of numbers are used: $\mathbb{N}$ for natural numbers, $\mathbb{Z}$ for integers, $\mathbb{Q}$ for rational numbers and $\mathbb{R}$ for real numbers.

Other symbols used in this work are listed in the following table, for those newly defined the explanation of symbol is supplemented with a respective page number.

$[i, j]$	closed interval in $\mathbb{R}$
(i, j)	open interval in $\mathbb{R}$
$\lfloor z \rfloor$	bottom integer part of the number $z \in \mathbb{R}$
$\{z\}$	fractional part of the number $z \in \mathbb{R}$
$:=$	equality giving a definition
$\doteq$	approximate equality
$<_{lex}$	relation "strictly less" on a lexicographically ordered set
$\Re z$	real part of a complex number z
$\Im z$	imaginary part of a complex number z
$\mathbb{Q}[\alpha]$	extension of the field $\mathbb{Q}$ by an irrational number α
$\mathbb{Z}[\lambda]$	ring of polynomials in λ with integer coefficients

$(x)_\beta$	β-expansion of a number x, *p. 9*
$\mathrm{Fin}(\beta)$	set of all numbers having finite β-expansion for given β, *p. 10*
$\mathrm{fp}(x)$	number of fractional digits in the β-expansion of x, *p. 10*
$d_\beta(1)$	Rényi development of unit, *p. 10*
$\mathbb{Z}_\beta$	set of all β-integers for given β, *p. 12*
$L_\oplus(\beta)$	maximal number of fractional digits arising under addition of two β-integers, *p. 17*
$L_\otimes(\beta)$	maximal number of fractional digits arising under multiplication of two β-integers, *p. 17*
$(x)_F$	representation of a number x in a n-generalized Fibonacci sequence, *p. 30*

Chapter 1

Introduction

In 1952 A. Rényi proved in his work *Representations for real numbers and their ergodic properties* [11], that for each real base $\beta > 1$ and for every positive real number x, there exists a unique representation by the expression

$$x = x_k\beta^k + x_{k-1}\beta^{k-1} + \ldots + x_0\beta^0 + x_{-1}\beta^{-1} + x_{-2}\beta^{-2} \ldots \tag{1.1}$$

which fulfills the conditions$x_i > 0$, $x - (x_k\beta^k + \ldots + x_i\beta^i) < \beta^i$ for $i = k, k-1, k-2, \ldots$ and where the coefficients $x_k, x_{k-1}, \ldots$ are non-negative integers less than β. The representation of the number x of this form is called the β-expansion of x.

For $\beta = 10$ or $\beta = 2$ the β-expansion of a number x is the ordinary representation of x in the decimal or binary numeration system. In these systems addition and multiplication of numbers with finite representation (1.1) is very simple and algorithms, which perform these arithmetic operations, as well as the number of required steps, are known. Similar algorithms are used generally in all systems where the base β is an integer > 1.

The situation is completely different if we start to count in a system with an irrational base β. In fact, only little is known about arithmetical rules in these systems. But why should we use these numeration systems at all?

Whether are the computations preformed by a human or by a computer, it is possible to work only with numbers that have a finite number of digits in their representation in the given numeration system (we will denote the set of these numbers having finite β-expansion by the symbol $\mathrm{Fin}(\beta)$). In a numeration system with an integer base β every number x with finite representation has the form $x = \frac{p}{q}$, where p, q are integers, i.e. x is a rational. If we choose for the base of our system an irrational number β, the set $\mathrm{Fin}(\beta)$ will contain also some irrational numbers x.

That is precisely why we concern ourselves with systems having irrational base: exact arithmetics with irrational numbers is the ground of the new methods for building aperiodic random numbers generators, for new cryptographic methods, and last but not least for mathematical modelling of the recently discovered materials with long range order – the so called quasicrystals.

This work is divided into three chapters.

The first chapter contains mathematical background necessary for this work. We remind basic notations from number theory such as algebraic numbers, Pisot numbers and algebraic extensions $\mathbb{Q}[\alpha]$ of the field $\mathbb{Q}$ of rationals. Then we introduce

β-expansion, i.e. representation numbers in a system with irrational basis, we show two ways to generate these representations and we define two important sets of numbers – the set of all integers in a system with irrational basis (i.e. β-integers) and previously mentioned set of all numbers having finite representation ($\mathrm{Fin}(\beta)$). We also demonstrate some interesting properties of the set of all β-integers. Finally there is indicated the main problem of the arithmetics of irrational numbers – the sum or product of two β-integers can have non-zero fractional part, the set $\mathrm{Fin}(\beta)$ even does not have to be closed under arithmetic operations.

The second chapter engages in the arithmetics of β-integers itself. At first we discuss some necessary and some sufficient conditions for $\mathrm{Fin}(\beta)$ to be a ring (i.e. be closed under arithmetic operations). The second section then describes a software library written for performing computations in the ring $\mathrm{Fin}(\beta)$ for some irrationalities β. Consecutively we show the way to generate the set of different β-integers large enough, needed in the last section of this chapter to obtain the lower bounds on $L_\oplus(\beta)$ and $L_\otimes(\beta)$ – i.e. on the number of fractional digits arising under addition and multiplication of β-integers – for some irrationalities β.

The last chapter describes two methods to estimate the maximal number of fractional digits that may appear by addition or multiplication of β-integers. These methods are then applied to three different numeration systems with irrational base (*minimal Pisot number numeration system, Tribonacci number numeration system and* γ, *where* γ *is the greatest root of the equation* $x^3 = 25x^2 + 15x + 2$).

Putting together the lower bounds obtained experimentally by means of computer computation in the second chapter and the upper bounds found theoretically using the properties of numbers, we gain either the exact values of $L_\oplus$ and $L_\otimes$ or possibly interval of values these coefficients can take.

This use of described methods includes fairly large amount of computations carried out by a computer which produce a lot of output data. These are written out in files contained on the associated CD-ROM. All programs written and used for these analysis are also on the CD-ROM. The content of the disc is summarized in one of the Appendixes, whereas the second one contains the user manual to the program BetaArithmetic.

All irrationalities to which we applied our methods for estimates on $L_\oplus$ and $L_\otimes$ were cubic algebraic integers. This is because for quadratic irrationalities enough results are known, especially for the golden mean, i.e. $\tau = \frac{1+\sqrt{5}}{2}$, the root of the equation $x^2 = x+1$ (see [8]).

In [5] there is shown that the values of coefficients are $L_\oplus(\beta) = L_\otimes(\beta) = 1$ for a β – root of an equation $x^2 = mx - 1$ and $L_\oplus(\beta) = L_\otimes(\beta) = 2$ for a β – root of an equation $x^2 = mx + 1$. More general formula can be found in [7] which asserts that for a $\beta > 1$ – root of an equation $x^2 = mx + n$, $m \le n$, the values of coefficients are bounded by following constants: $2\left\lfloor\frac{m+1}{m-m+1}\right\rfloor \le L_\oplus \le 2\left\lceil\frac{m}{m-n+1}\right\rceil$ and $L_\otimes \le 4L_\oplus \log_2(m+2)$.

The only known results relating to cubic irrationalities are the estimates for the β - Tribonacci number. In [9] there is shown that $L_\otimes(\beta) \le 9$ and Arnoux conjectures there that $L_\otimes(\beta) = 3$. One of our results refutes conjecture of Arnoux and improves the bound found by Messaoudi. We show that $4 \le L_\otimes \le 5$.

Chapter 2

Preliminaries

This first chapter summarizes the mathematical background necessary for the whole work.

In the first section we remind the some basic terms from the number theory: algebraic numbers, Pisot numbers and corresponding extensions $\mathbb{Q}[\alpha]$ of the field $\mathbb{Q}$ by an irrational number α.

The second section introduces way to represent real numbers in the numeration system with an irrational base. There are two ways for generating of these representations and also some commonly used terms connected with them. Two examples (*Tribonacci numeration system* and *minimal Pisot numeration system*) are introduced to demonstrate preceding notion.

Third part demonstrates some interesting properties of sets of all integers in systems with irrational base, impossible to see in any system with integer base, whereas the last section contains short remark about unusual but very important properties of arithmetic operations in these systems.

2.1 Algebraic numbers

Definition 2.1.1. *The number $x \in \mathbb{C}$ is called algebraic number, if it is a root of the polynomial*

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + x_0, \qquad a_0, \ldots, a_n \in \mathbb{Z} \tag{2.1}$$

Definition 2.1.2. *A non-zero polynomial of the form* (2.1) *with the minimal possible degree, having an algebraic number α as its root, is called the minimal polynomial of α, the degree of this polynomial is called the degree of α.*

Obviously the minimal polynomial $p(x)$ of an algebraic number α is irreducible over the field $\mathbb{Q}$ (otherwise there exists some polynomial of smaller degree with a root α, which is the contradiction with the minimality of $p(x)$) and every root of the polynomial $p(x)$ has the multiplicity equal to one (the same argumentation).

Definition 2.1.3. *Let α be an algebraic number of the degree m. Let us denote $\alpha^{(1)} := \alpha$ and the other roots of the minimal polynomial $\alpha^{(2)}, \ldots, \alpha^{(m)}$. These numbers are called Galois conjugates of the number α.*

It is well known, that the minimal subfield of complex numbers $\mathbb{C}$ which contains all the rationals $\mathbb{Q}$ as well as the algebraic number α of the degree m is

$$\mathbb{Q}[\alpha] \equiv \mathbb{Q}[\alpha^{(1)}] := \{y_0 + y_1\alpha + y_2\alpha^2 + \ldots + y_{m-1}\alpha^{m-1} \mid y_i \in \mathbb{Q}, \forall i = 0, \ldots m-1\}.$$

Identically we can construct these minimal fields for all the Galois conjugates of the number α: $\mathbb{Q}[\alpha^{(2)}], \ldots, \mathbb{Q}[\alpha^{(m)}]$. These fields are mutually isomorphic. The isomorphism (so called *k-th Galois isomorphism*) $\mathbb{Q}[\alpha] \to \mathbb{Q}[\alpha^{(k)}]$ is induced by the assignment $\alpha \mapsto \alpha^{(k)}$, hence for $\beta \in \mathbb{Q}[\alpha]$ the image $\beta^{(k)}$ under the k-th Galois isomorphism is

$$\beta = y_0 + y_1\alpha + \ldots + y_{m-1}\alpha^{m-1} \mapsto \beta^{(k)} = y_0 + y_1\alpha^{(k)} + \ldots + y_{m-1}(\alpha^{(k)})^{m-1}.$$

For the numbers $\beta, \gamma \in \mathbb{Q}[\alpha]$, the k-th Galois isomorphism fulfills following relations

$$(\beta+\gamma)^{(k)} = \beta^{(k)} + \gamma^{(k)}$$
$$(\beta.\gamma)^{(k)} = \beta^{(k)}.\gamma^{(k)}.$$

Definition 2.1.4. *An algebraic number α of the order m with monic minimal polynomial (i.e. the leading coefficient a_m is equal to 1) is called an algebraic integer. An algebraic integer is said to be a Pisot number if it is greater than 1 and all its Galois conjugates have modulus strictly less than one. Pisot number is called unitary if the product of all roots of its minimal polynomial is +1 or -1.*

2.2 Beta expansions

Definition 2.2.1. *Let $\beta > 1$ be a real number. A β-representation of a number $x \in \mathbb{R}_+$ is a sequence $(x_i)_{k \geq i \geq -\infty}$ of non-negative integers, such that*

$$x = x_k\beta^k + x_{k-1}\beta^{k-1} + \ldots + x_1\beta + x_0 + x_{-1}\beta^{-1} + \ldots \tag{2.2}$$

We use usual radix scale for a β-representation of a number x

$$(x)_\beta = x_k x_{k-1} \ldots x_1 x_0 \bullet x_{-1} x_{-2} \ldots$$

the symbol between coefficients x_0 and x_{-1} is called *fractional point*.

Certainly, there can be more than one β-representation of a given $x \in \mathbb{R}_+$.

Example. *Let β be the real root of the equation $x^3 = x^2 + x + 1$. Since $2\beta^3 + \beta = \beta^3 + \beta^2 + 2\beta + 1$, the sequences*

$$2010 \bullet 000\ldots \qquad \text{and} \qquad 1121 \bullet 000\ldots$$

are the β-representations of the same number x.

Among all β-representations of a given $x \in \mathbb{R}_+$ there exists one particular β-representation – called *β-expansion*.

Definition 2.2.2. *The β-expansion of a non-negative number x is the sequence $(x_i)_{k \geq i > -\infty}$ computed by the "greedy algorithm":*

1. *find $k \in \mathbb{Z}$, such that $\beta^k \le x < \beta^{k+1}$*
2. $x_k := \lfloor x/\beta^k \rfloor$, $r_k := \{x/\beta^k\}$
3. $i := k - 1$
4. $x_i := \lfloor \beta r_{i+1} \rfloor$, $r_i = \{\beta r_{i+1}\}$
5. *if $r_i \neq 0$ then $i := i - 1$ and go to step 4.*

The described algorithm implies:

1. $x_i \in \{0, 1, \ldots, \lceil \beta \rceil - 1\}$
2. $0 < x - x_k\beta^k - x_{k-1}\beta^{k-1} - \ldots - x_i\beta^i < \beta^i$.

Since $\lim_{i \to -\infty} \beta^i = 0$, the sequence gained by this algorithm is a β-representation.

Example. *Let β and x are the same as in the last example (i.e. $\beta \doteq 1,83928676$, $x \doteq 14,283812$), we can find β-expansion of the number x using "greedy algorithm". The highest power of β less than x is β^4. By the sequential calculation we get the result*

$$10011 \bullet 000 \ldots$$

If an expansion ends in infinitely many zeros, is is said to be finite and the ending zeros are omitted. The set of all real numbers x, for which the β-expansions of their modulus $|x|$ is finite, is denoted by

$$\mathrm{Fin}(\beta) := \{x \in \mathbb{R} \mid |x| \text{ has a finite } \beta\text{-expansion}\}.$$

Definition 2.2.3. *Let $x \in \mathbb{R}$, $x > 0$ and let $\sum_{k=-N}^{n} x_k\beta^k$ be its β-expansion with $x_{-N} \neq 0$. If $N > 0$ the $r = \sum_{k=-N}^{-1} x_k\beta^k$ is called the β-fractional part of x. If $N \le 0$ we set $\mathrm{fp}(x) := 0$, for $N > 0$ we define $\mathrm{fp}(x) := N$, i.e. $\mathrm{fp}(x)$ is the number of fractional digits in the β-expansion of x.*

For $0 \le x < 1$ the coefficients of the β-expansion of x may be expressed using the β-transformation of the unit interval

$$T_\beta(x) := \{\beta x\}, \quad x \in [0, 1)$$

and formula

$$x_{-k-1} = \lfloor \beta T_\beta^k(x) \rfloor. \tag{2.3}$$

For $x \in (0, 1)$ the coefficients x_{-k-1} coincide with the coefficients obtained by the greedy algorithm. For $x = 1$ the output of the greedy algorithm is $1 = 1\bullet$, whereas the formula (2.3) gives the *Rényi development of 1* $d_\beta(1)$

$$d_\beta(1) = t_1 t_2 \ldots, \quad \text{where} \quad t_k = \lfloor \beta T_\beta^{k-1}(1) \rfloor.$$

Obviously, the numbers t_k are non-negative integers smaller than β, $t_1 = \lfloor \beta \rfloor$ and

$$1 = \sum_{k=1}^{\infty} t_k \beta^{-k}. \tag{2.4}$$

Using the greedy algorithm we can find a unique β-expansion for every $x \in \mathbb{R}_+$. However, there exist sequences (2.2) that do not correspond to any number x. The first condition for a β-representation to be a β-expansion follows from the greedy algorithm: the coefficients in the expansion are the non-negative integers less or equal to $\beta - 1$ for $\beta \in \mathbb{N} - \{1\}$, or $\lfloor \beta \rfloor$ for $\beta \notin \mathbb{N},\ \beta > 1$[1] i.e. the coefficients are less or equal to $\lceil \beta \rceil - 1$.

One can imagine a β-expansion of a number x as a infinite word in an alphabet $\mathcal{A} = \{0, 1, \ldots, \lceil \beta \rceil - 1\}$ with a marked position of the fractional point. For $\beta \in \mathbb{N}$, $\beta > 1$ every word in this alphabet corresponds to a β-expansion of some number x. For $\beta \notin \mathbb{N}$ is the situation different. It is clear that the answer to a question whether a infinite word in the alphabet $\mathcal{A}$ corresponds to a β-expansion of a number x does not depend on the position of the fractional point. Therefore we will consider words $s_1 s_2 s_3 \ldots$ and interpret their value as $x = \bullet s_1 s_2 \ldots = s_1 \beta^{-1} + s_2 \beta^{-2} + \cdots$. In order to state necessary and sufficient condition (see [10]) for a word $s_1 s_2 s_3 \ldots$ to be a β-expansion of a number x we need following definitions.

Definition 2.2.4. *Let $\boldsymbol{s} = s_1 s_2 s_3 \ldots s_n \ldots$ be an infinite sequence of non-negative numbers. The sequence $\sigma(\boldsymbol{s}) = s_2 s_3 \ldots s_n \ldots$ is called the shift of sequence $\boldsymbol{s}$. The symbol $\sigma^p(\boldsymbol{s})$ denotes p-fold multiple shift, i.e. p shifts applied to the sequence $\boldsymbol{s}$ stepwise.*

Definition 2.2.5. *Let x be a finite word. The symbol x^ω denotes the sequence $xxx\ldots$, i.e. infinite-fold multiple concatenation of the word x to itself.*

Theorem 2.2.1 (Parry). *Let β be a real number strictly greater than one. Let $d_\beta(1) = t_1 t_2 \ldots$ be the Rényi development of 1. Let $\boldsymbol{s}$ be an infinite sequence of non-negative integers.*

(i) If $d_\beta(1)$ is infinite, the condition

$$\forall p \geq 0 \quad \sigma^p(\boldsymbol{s}) <_{lex} d_\beta(1)$$

is necessary and sufficient for the sequence $\boldsymbol{s}$ to be the β-expansion of some $x \in [0, 1)$.

(ii) If $d_\beta(1)$ is finite, say $d_\beta(1) = t_1 \ldots t_{n-1} t_n$, then $\boldsymbol{s}$ is the β-expansion of a number $x \in [0, 1)$ if and only if

$$\forall p \geq 0 \quad \sigma^p(\boldsymbol{s}) <_{lex} d^*(1, \beta) = (t_1 \ldots t_{n-1}(t_n - 1))^\omega.$$

Notice that the finite sequence $c_n c_{n-1} \ldots c_0$ of non-negative integers satisfies the Parry condition above, if

$$c_i c_{i-1} \ldots c_0 <_{\text{lex}} d_\beta(1) \qquad \forall i = 0, 1, \ldots, n$$

Such sequences (words) are called *admissible*. A finite sequence of non-negative integers is called *forbidden*, if it is not admissible.

[1]These two conditions can be replaced by the only one: the coefficients are non-negative integers less or equal to $\lceil \beta \rceil - 1$.

For s given base β, an important role is played by those numbers play these ones, whose β-expansion has no fractional part. These numbers are called *β-integers*, the set of all β-integers for given β is denoted by $\mathbb{Z}_\beta$.

$$\mathbb{Z}_\beta := \left\{ x \in Fin(\beta) \;\middle|\; |x| = \sum_{i=0}^{k} x_i \beta^i \text{ is the } \beta\text{-expansion} \right\}.$$

If β is as integer, $\beta \geq 2$, the set $\mathbb{Z}_\beta$ coincides with the set of all rational integers $\mathbb{Z}$.

2.3 Examples of numeration systems

2.3.1 Minimal Pisot number numeration system

The base of this numeration system is the minimal Pisot number. It is known that the smallest among all irrational Pisot numbers is the real number β, solution of the equation $x^3 = x + 1$.

- The value of the base is $\beta \doteq 1.325$, hence only 0 and 1 are allowed to be coefficients os the β-expansions.

- Two other roots of this equation are complex numbers $\beta' \doteq -0.662 + i0.562$ and $\beta'' \doteq -0.662 - i0.562$. They are mutually complex conjugated, their modulus is $|\beta'| = |\beta''| \doteq 0.869$.

- The Rényi development of unit is $d_\beta(1) = 10001$. Therefore an infinite word $x_k \dots x_0 \bullet x_{-1} x_{-2} \dots$ over the alphabet $\mathcal{A} = \{0, 1\}$ is a β-expansion in the minimal Pisot numeration system, if and only if digits 1 are separated by at least four zeros, i.e. $x_i + x_{i-1} + \dots + x_{i-4} \leq 1$ for $i \leq k$.

2.3.2 Tribonacci numeration system

Second example of a numeration system with an irrational base β is the Tribonaci numeration system. The base of this system is the so-called Tribonacci number – the real root of the equation $x^3 = x^2 + x + 1$.

- The value of the base is $\beta \doteq 1.839$, hence the β-expansions have coefficients only 0 and 1.

- Two other roots of this equation are complex numbers $\beta' \doteq -0.419 + i0.606$ and $\beta'' \doteq -0.419 - i0.606$. Obviously, they are mutually complex conjugated, their modulus is $|\beta'| = |\beta''| \doteq 0.737$.

- The Rényi development of 1 in this case is $d_\beta(1) = 111$, therefore the forbidden string of coefficients is 111, i.e. behind each pair of 1 there has to be at least one 0 in any β-expansion in this numeration system. More precisely if $x_k \dots x_0 \bullet x_{-1} x_{-2} \dots$ is a β-expansion then $x_i x_{i-1} x_{i-2} = 0$ for $i \leq k$.

Remark. Tribonacci number from is a special case of the so-called *n-generalized golden mean*, i.e. the real root of the equation

$$x^n = x^{n-1} + \cdots + x + 1 .$$

We will demonstrate two interesting properties of the set $\mathbb{Z}_\beta$ on a example of n-generalized golden mean in the section 2.4.

2.3.3 Numeration system with base β solution of the equation $x^3 = 25x^2 + 15x + 2$

The Pisot number β, which is the base of this numeration system, is the greatest root of the equation $x^3 = 25x^2 + 15x + 2$.

- The value of the base is $\beta \doteq 25.589$. Since $\lfloor \beta \rfloor = 25$, β-expansion of any x is a word in the 26 letter alphabet $\mathcal{A}$. To avoid confusion we will denote the letters of this alphabet as $\mathcal{A} = \{(0), (1), \ldots, (25)\}$.
- Two other roots of the equation are $\beta' \doteq -0.388$ and $\beta'' \doteq -0.202$.
- Rényi development on unit is $d_\beta(1) = (25)(15)(2)$, therefore behind each coefficient (25) there has to be either a coefficient smaller than 15 or one of the words (15)(0), (15)(1).
- Note that this Pisot number differ from all previous examples: it is not a unit and it is totally real.

2.4 The set of beta integers $\mathbb{Z}_\beta$

If $\beta \in \mathbb{N}$ then $\mathbb{Z}_\beta = \mathbb{Z}$ and therefore all distances between neighbours in the set $\mathbb{Z}_\beta$ are equal to 1. For $\beta \notin \mathbb{N}$, the constituent neighbouring elements of the set $\mathbb{Z}_\beta$ are not well-proportioned in the real axis, contrariwise there are more admissible distances between neighbours (see Figure 2.1). The set of distances between neighbours in $\mathbb{Z}_\beta$ has at least two elements.

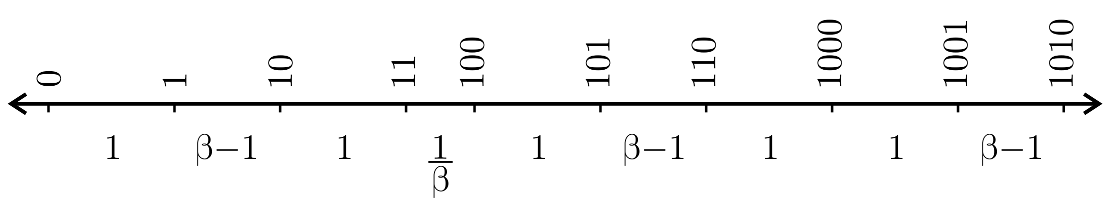

Figure 2.1: Several elements of the set $\mathbb{Z}_\beta$ in the Tribonaci numeration system

Theorem 2.4.1. [13]. *Let $d_\beta(1) = t_1 t_2 \ldots$ be the Rényi development of unit. Then $D > 0$ is a distance between neighbours in the set $\mathbb{Z}_\beta$ if and only if*

$$D = \sum_{i=0}^{\infty} \frac{t_{i+k}}{\beta^{i+1}} , \qquad \textit{for some } k = 1, 2, \ldots$$

It is easy to see that the set of distances between neighbours in $\mathbb{Z}_\beta$ is finite if and only if the Rényi development of unit is finite or eventually periodic, numbers β fulfilling this condition are called β-numbers.

We will state and prove previous theorem for a special irrationality – n-generalized golden mean.

Proposition 2.4.2. *Let β be the n-generalized golden mean. Then there are just n different distances between neighbouring elements of the set $\mathbb{Z}_\beta$, namely*

$$D_i = \sum_{i=1}^{k} \frac{1}{\beta^i}, \quad \textit{for } k = 1, 2, \ldots, n.$$

Proof. Let x and y be the neighbouring elements of the set $\mathbb{Z}_\beta$. We divide the proof into two parts.

1. Let x and y have β-expansions of different lengths. Let x be the smaller one and y the larger one. Let k be the length of x.

   As y is longer than x, its β-expansion begins with $\alpha_{k+1} = 1$ and the other coefficients in the expansion are zeros – if there were some non-zero coefficient, we could replace it with zero and so create the β-expansion of number w: $x < w < y$, which is in contradiction with the condition that x and y are neighbours.

   By the same token the β-expansion of x is as dense (full of 1s) as possible, starting with $\alpha_k = 1$.

   (a) Let $0 \le k < n$. As the β-expansion of x is shorter than the length of forbidden factor $d_\beta(1)$, all its coefficients can be equal to 1. The difference between numbers then depends on k[2]:

   $$1\underbrace{0\ldots0}_{|.|=k} - \underbrace{1\ldots1}_{|.|=k} = 0.\underbrace{1\ldots1}_{|.|=n-k}$$

   Therefore if $d_\beta(1) = t_1 \ldots t_n$ is the Rényi development of 1, all the elements of the set $\{0.t_{n-i} \ldots t_n \mid 0 \le i < n\}$ are the distances between neighbouring elements in the set $\mathbb{Z}_\beta$.

   (b) Let $n \le k$. In this case the β-expansion of x is longer than n, hence not all its coefficients can be equal to 1; since it has to be dense, it will be composed of as many factors $1\ldots10$ of length n as possible followed by factor $1\ldots1$ of length less than n.

   It is easy to see that the results of subtraction will be the same as for $0 \le k < n$. Let us show one example for $n = 4$:

   $$\begin{aligned} &1\ 0000\ 0000\ 0000\ 00 - 1110\ 1110\ 1110\ 11 = \\ &\qquad = 1110\ 1110\ 1110\ 11.11 - 1110\ 1110\ 1110\ 11 = 0.11 \end{aligned}$$

[2]the symbol $|.|$ denotes the length of respective string

2. Let x and y have β-expansions of the same length (e.g. k). Let l be the smallest number such that $\alpha^{(y)}_{k-1-l} \neq \alpha^{(x)}_{k-1-l}$, where $\alpha^{(y)}_i$ and $\alpha^{(x)}_i$ are coefficients in the β-expansions of y and x respectively. It means that $\alpha^{(y)}_i = \alpha^{(x)}_i \ \forall i \in \{k-1, k-2, \dots, k-l\}$, these coefficient has no effect on the result of subtraction and we can discard them. After this discard the numbers will have β-expansions of different lengths. We have already solved this case.

$\square$

The distances between β-integers have one more important property. Let us denote constituent distances by letters

$$
\begin{aligned}
A^{(0)} &= .t_1 \dots t_n \\
A^{(1)} &= .t_2 \dots t_n \\
&\vdots \\
A^{(n-1)} &= .t_n
\end{aligned}
$$

From the proof of Proposition 2.4.2 easily follows, that the gap $A^{(i)}$ is between two neighbouring β-integers if and only if the expansion of the shorter one ends in arbitrary (e.g. l) number of factors $1 \dots 10$ of length n with the factor $1 \dots 1$ of length i on the very end and the expansion of the longer one ends with the number 1 followed by $l * n + i$ zeros.

Let us take a pair of numbers, which forms some of the gaps $A^{(0)}, \dots, A^{(n-1)}$ and multiply both numbers by β. These multiplied numbers will certainly remain the elements of the set $\mathbb{Z}_\beta$. We concern ourselves with the question whether they are immediate neighbours in the lexicographically ordered set $\mathbb{Z}_\beta$ too.

1. Let $i < n-1$. Let $x, y \in \mathbb{Z}_\beta$ be the neighbours with the gap $A^{(i)}$ between them. x and y multiplied by β

$$
\begin{aligned}
\beta\mathrm{x} &= 0\,\overbrace{11\dots10}^{|.|=n}\dots\overbrace{11\dots10}^{|.|=n}\overbrace{11\dots1}^{|.|=i}\,0 \\
\beta\mathrm{y} &= 1\,\underbrace{00\dots00\dots00\dots00\,00\dots0}_{|.|=l*n+i}\,0
\end{aligned}
$$

are surely not the neighbours, because there is the number

$$
\mathrm{w} := 0\,\overbrace{11\dots10}^{|.|=n}\dots\overbrace{11\dots10}^{|.|=n}\overbrace{11\dots1}^{|.|=i}\,1
$$

between them. The distances are $|w - \beta x| = 1 = A^{(0)}$ and $|\beta y - w| = A^{(i+1)}$. Hence we gain following substitution rules

$$
A^{(i)} \mapsto A^{(0)} A^{(i+1)} \quad \forall i \in [0, n-1)\,. \tag{2.5}
$$

2. Let $x, y \in \mathbb{Z}_\beta$ be the neighbours with the gap $A^{(n-1)}$ between them. In this case the β-multiples of x and y

$$\beta\text{x} = 0\,\overbrace{11\ldots10}^{|.|=n}\ldots\overbrace{11\ldots10}^{|.|=n}\overbrace{11\ldots10}^{|.|=n}$$
$$\beta\text{y} = 1\,\underbrace{00\ldots00\ldots00\ldots0000\ldots0}_{|.|=l*n+n}\,0$$

remains the neighbours in the set $\mathbb{Z}_\beta$. The gap $A^{(}n-1)$ between them was transformed into $A^{(0)}$. Therefore we gain one more substitution rule

$$A^{(n-1)} \mapsto A^{(0)}\,. \tag{2.6}$$

As we have discussed all possible gaps among elements of the set $\mathbb{Z}_\beta$, we can conclude that using obtained substitutions (2.5) and (2.6) we are able to easily generate the whole set $\mathbb{Z}_\beta$ (for several elements of the set $\mathbb{Z}_\beta$ is this procedure graphically demonstrated on the Figure 2.2).

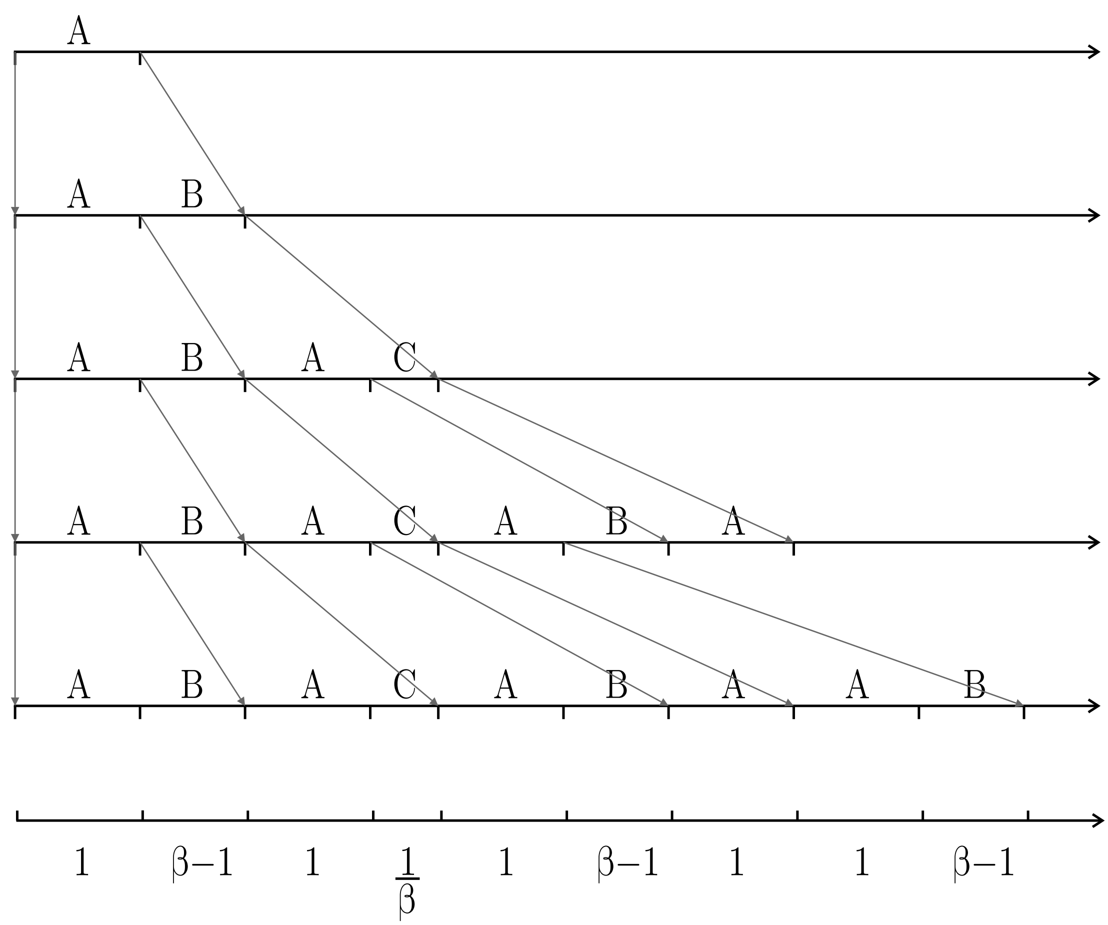

Figure 2.2: Generation of the set $\mathbb{Z}_\beta$ for β Tribonacci number, by means of substitutions

2.5 Arithmetic properties of beta-expansions

In case that $\beta \in \mathbb{Z}$, $\beta > 1$, $\mathrm{Fin}(\beta)$ is closed under operations of addition, subtraction and multiplication, i.e. $\mathrm{Fin}(\beta)$ is a ring. It is also easy to determine the β-expansion of $x+y$, $x-y$ and $x \cdot y$ with the knowledge of the β-expansions of x and y. The result of these arithmetic operations for two β-integers is always a β-integer.

In case that $\beta > 1$ is not a rational integer, the situation is more complicated and generally we do not know any criteria which would decide whether $\mathrm{Fin}(\beta)$ is a ring or not. Unlike the integer base, the result of addition, subtraction and multiplication of two β-integers can have non-zero fractional part.

Example. *We will sum the numbers 1 and 1 in the Tribonaci numeration system. The value of the result is 2. But the number 2 is forbidden coefficient in the β-expansion of any number. Thus in this numeration system We have to use another technique: we substitute one 1 with its numerical equivalent – the Rényi development of unit. Only then we sum the numbers.*

$$1 + 1 = 1 + 0.111 = 1.111$$

However the result is not yet admissible, because it contains the forbidden factor 111. Now we substitute the most left occurrence of the factor 0.111 with the word 1.

$$1.111 \mapsto 10.001$$

Finally, this is the correct β-expansion of the number 2 in the Tribonaci numeration system.

Besides the question whether results of these arithmetic operations are always finite or not, we are also interested in describing the length of the resulting β-fractional part. It is possible to convert the sum $x + y$ and the product $x \cdot y$ of two numbers $x, y \in \mathrm{Fin}(\beta)$ by multiplication by a suitable factor β^k into a sum or product of two β-integers. Therefore we define following constants.

Definition 2.5.1. *Let $\beta > 1$. We denote*

$$\begin{aligned} L_\oplus = L_\oplus(\beta) &:= \max\{\mathrm{fp}(x+y) \mid x, y \in \mathbb{Z}_\beta,\ x+y \in \mathrm{Fin}(\beta)\} \\ L_\otimes = L_\otimes(\beta) &:= \max\{\mathrm{fp}(x \cdot y) \mid x, y \in \mathbb{Z}_\beta,\ x \cdot y \in \mathrm{Fin}(\beta)\}. \end{aligned}$$

Chapter 3

Algorithms for the arithmetics

In this chapter we at first discuss the question of the set $\mathrm{Fin}(\beta)$ being a ring. We prove that it is enough to study the question of addition of positive numbers to decide whether $\mathrm{Fin}(\beta)$ is or is not a ring. Afterwards we show necessary and sufficient condition for $\mathrm{Fin}(\beta)$ being closed under addition of positive numbers.

In the prove of the theorem stating the sufficient condition, we show the algorithm for performing arithmetic operations. This algorithm was used to write a computer program BetaArithmetic. This program is in detail described in the second section of this chapter.

The third section engages in a problem of consecutive generation of all β-integers in a numeration system with irrational base. This is a crucial task when we try to find good lower bounds on the values of $L_\oplus$ and $L_\otimes$. In the last section of this chapter, there are found some lower bounds on these coefficients, namely for those numeration systems, introduced in the second chapter.

3.1 Addition of positive β-integers

According to our definition from the previous section, $\mathrm{Fin}(\beta)$ contains both positive and negative numbers. Therefore we first justify why, in order to decide about $\mathrm{Fin}(\beta)$ being a ring, we shall study only the question of addition of positive numbers.

Proposition 3.1.1. *Let $\beta > 1$ and $d_\beta(1)$ be its development of unit.*

(i) If $d_\beta(1)$ is infinite, then $\mathrm{Fin}(\beta)$ *is not a ring.*

(ii) If $d_\beta(1)$ is finite, then $\mathrm{Fin}(\beta)$ *is a ring if and only if* $\mathrm{Fin}(\beta)$ *is closed under addition of positive elements.*

Proof. (i) Let $d_\beta(1) = t_1 t_2 t_3 \dots$ be infinite. Then (2.4) implies

$$1 - \frac{1}{\beta} = \frac{t_1 - 1}{\beta} + \frac{t_2}{\beta^2} + \frac{t_3}{\beta^3} + \cdots \tag{3.1}$$

Since $(t_1 - 1)t_2 t_3 \dots <_{lex} d_\beta(1)$, the expression on the right hand side of (3.1) is the β-expansion of $1 - \beta^{-1}$ which therefore does not belong to $\mathrm{Fin}(\beta)$.

(ii) Let

$$1 = \frac{t_1}{\beta} + \frac{t_2}{\beta^2} + \cdots + \frac{t_m}{\beta^m}, \tag{3.2}$$

and let $\mathrm{Fin}(\beta)$ be closed under addition of positive numbers. Consider arbitrary $x \in \mathrm{Fin}(\beta)$ and arbitrary $\ell \in \mathbb{Z}$ such that $x > \beta^\ell$. Then the β-expansion of x has the form

$$x = \sum_{i=-N}^{n} x_i \beta^i$$

where $n \geq \ell$. Repeated application of (3.2) allows us to create another representation of x, say

$$x = \sum_{i=-M}^{\ell} \tilde{x}_i \beta^i$$

such that $\tilde{x}_\ell \geq 1$. Then

$$(\tilde{x}_\ell - 1)\beta^\ell + \sum_{i=-M}^{\ell-1} \tilde{x}_i \beta^i$$

is a finite β-representation of $x - \beta^\ell$. Such representation can be interpreted as a sum of a finite number of positive elements of $\mathrm{Fin}(\beta)$, which is, according to the assumption, again in $\mathrm{Fin}(\beta)$.

It suffices to realize that subtraction $x - y$ of arbitrary $x, y \in \mathrm{Fin}(\beta)$, $x > y > 0$ is a finite number of subtractions of some powers of β. Therefore $\mathrm{Fin}(\beta)$ being closed under addition of positive elements implies being closed under addition of arbitrary $x, y \in \mathrm{Fin}(\beta)$.

Since multiplication of numbers $x, y \in \mathrm{Fin}(\beta)$ is by the distributive law addition of a finite number of summands from $\mathrm{Fin}(\beta)$, the proposition is proved. □

From now on, we focus on addition $x + y$ for $x, y \in \mathrm{Fin}(\beta)$, $x, y \geq 0$. Without loss of generality we can assume $x, y \in \mathbb{Z}_\beta$.

Let $x, y \in \mathbb{Z}_\beta$, $x, y \geq 0$ with β-expansions

$$x = \sum_{k=0}^{n} a_k \beta^k \quad \text{and} \quad y = \sum_{k=0}^{n} b_k \beta^k.$$

By simple summation of corresponding coefficients we obtain a β-representation of the sum $x + y$

$$\sum_{k=0}^{n} (a_k + b_k)\beta^k.$$

If the sequence of coefficients $(a_n + b_n)(a_{n-1} + b_{n-1}) \ldots (a_0 + b_0)$ verifies the Parry condition (Theorem 2.2.1), we have directly the β-expansion of $x + y$. In the opposite case, the sequence must contain a forbidden factor.

Special role in our consideration play the so-called *minimal forbidden strings*.

Definition 3.1.1. *Let $\beta > 1$. A forbidden string $u_k u_{k-1} \ldots u_0$ of non-negative integers is called minimal, if* *(i)* $u_{k-1}\dots u_0$ *and* $u_k \dots u_1$ *are admissible, and*

(ii) $u_i \geq 1$ *implies* $u_k \dots u_{i+1}(u_i - 1)u_{i-1}\dots u_0$ *is admissible, for all* $i = 0, 1, \dots, k$.

Obviously, a minimal forbidden string $u_k u_{k-1} \dots u_0$ contains at least one non-zero digit, say $u_i \geq 1$. The string is a β-representation of the addition of two β-integers

$$z = u_k\beta^k + \cdots + u_{i+1}\beta^{i+1} + (u_i - 1)\beta^i + u_{i-1}\beta^{i-1} + \cdots + u_0$$
$$w = \beta^i.$$

The β-expansion of a number is lexicographically the greatest among all its β-representations, and thus if the sum $z + w$ belongs to $\mathrm{Fin}(\beta)$, then there exists a finite β-representation of $z + w$ lexicographically strictly greater than $u_k u_{k-1} \dots u_0$, (the β-expansion of $z + w$).
We have thus shown the following necessary condition:

Proposition 3.1.2 (Property T). *If* $\mathrm{Fin}(\beta)$ *is closed under addition of two positive numbers, then* β *must satisfy the following property:*

For every minimal forbidden string $u_k u_{k-1} \dots u_0$ *there exists a finite sequence of non-negative integers* $v_n v_{n-1} \dots v_\ell$ *, such that*

1. $k, \ell \leq n$,
2. $v_n\beta^n + \cdots + v_\ell\beta^\ell = u_k\beta^k + \cdots + u_1\beta + u_0$,
3. $v_n v_{n-1} \dots v_\ell >_{lex} \underbrace{00\dots0}_{(n-k)times} u_k \dots u_0$.

The rewriting of the β-representation $z = u_k\beta^k + \cdots + u_1\beta + u_0$ on a lexicographically strictly greater β-representation $z = v_n\beta^n + \cdots + v_\ell\beta^\ell$ will be called a transcription.

In general we shall apply the transcription on a β-representation of a number z in the following way. Every β-representation of z which contains a forbidden string can be written as a sum of a minimal forbidden string $\beta^j(u_k\beta^k + \cdots + u_1\beta + u_0)$ and a β-representation of some number $\tilde{z}$. The new transcribed β-representation of z is obtained by digit-wise addition of the transcription $\beta^j(v_n\beta^n + \cdots + v_\ell\beta^\ell)$ of the minimal forbidden string and the β-representation of $\tilde{z}$.

Obviously, the transcribed β-representation of z is lexicographically strictly greater than the original one. This transcription may be repeated until the β-representation does not contain any forbidden string. In general, it can happen that the procedure may be repeated infinitely many times. The following theorem provide sufficient condition, in order that this situation is avoided.

Theorem 3.1.3. *Let* $\beta > 1$ *satisfy Property T, and suppose that for every minimal forbidden string* $u_k u_{k-1} \dots u_0$ *we have the following condition:*

If $v_n v_{n-1} \dots v_\ell$ *is the lexicographically greater string of T corresponding to* $u_k u_{k-1} \dots u_0$ *then*

$$v_n + v_{n-1} + \cdots + v_l \leq u_k + u_{k-1} + \cdots + u_0\,.$$

Then $\mathrm{Fin}(\beta)$ *is closed under addition of positive elements. Moreover, for every positive* $x, y \in \mathrm{Fin}(\beta)$, *the* β*-expansion of* $x + y$ *can be obtained from any* β*-representation of* $x + y$ *using finitely many transcriptions.*

Proof. Without loss of generality, it suffices to decide about finiteness of the sum $x+y$ where $x = \sum_{k=0}^{n} a_k\beta^k$ and $y = \sum_{k=0}^{n} b_k\beta^k$ are the β-expansions of x and y, respectively. We prove the theorem by contradiction, i.e. suppose that we can apply a transcription to the β-representation $\sum_{k=0}^{n}(a_k + b_k)\beta^k$ infinitely many times.

We find $M \in \mathbb{N}$ such that $x + y < \beta^{M+1}$. Then the β-representation of $x + y$ obtained after the k-th transcription is of the form

$$x + y = \sum_{i=\ell_k}^{M} c_i^{(k)}\beta^i \,,$$

where ℓ_k is the smallest index of non-zero coefficient in the β-representation in the k-th step.

Since for every exponent $i \in \mathbb{Z}$ there exists a non-negative integer f_i such that $x + y \le f_i\beta^i$, we have that $c_i^{(k)} \le f_i$ for every step k.

Realize that for every index $p \in \mathbb{Z}$, $p \le M$, there are only finitely many sequences $c_M c_{M-1} \dots c_p$ satisfying $0 \le c_i \le f_i$ for all $i = M, M-1, \dots, p$. Since in every step k the sequence $c_M^{(k)} c_{M-1}^{(k)} \dots$ lexicographically increases, we can find for every index p the step κ, so that the digits $c_M^{(k)}$, $c_{M-1}^{(k)}$, $\dots$, $c_p^{(k)}$ are constant for $k \ge \kappa$. Formally, we have

$$(\forall p \in \mathbb{Z},\, p \le M)(\exists \kappa \in \mathbb{N})(\forall k \in \mathbb{N},\, k \ge \kappa)(\forall i \in \mathbb{Z},\, M \ge i \ge p)(c_i^{(k)} = c_i^{(\kappa)}) \qquad (3.3)$$

Since by assumption of the proof, the transcription can be performed infinitely many times, it is not possible that the digits $c_i^{(\kappa)}$ for $i < p$ are all equal to 0. Let us denote by t the maximal index $t < p$ with non-zero digit, i.e. $c_t^{\kappa} \ge 1$.

In order to obtain the contradiction, we use the above idea (3.3) repeatedly. For $p = 0$ we find $\kappa =: \kappa_1$ and $t =: t_1$ satisfying

$$x + y = \sum_{i=0}^{M} c_i^{(\kappa_1)}\beta^i + c_{t_1}^{(\kappa_1)}\beta^{t_1} + \sum_{i=\ell_{\kappa_1}}^{t_1-1} c_i^{(\kappa_1)}\beta^i \,.$$

In further steps $k \ge \kappa_1$ the digit sum $\sum_{i=0}^{M} c_i^{(k)}$ remains constant, since the digits $c_i^{(k)}$ remain constant. The digit sum $\sum_{i=t_1}^{(-1)} c_i^{(k)} \ge 1$, because the sequence of digits lexicographically increases. For every $k \ge \kappa_1$ we therefore have

$$\sum_{i=t_1}^{M} c_i^{(k)} \quad \ge \quad 1 + \sum_{i=0}^{M} c_i^{(k)} \,.$$

We repeat the same considerations for $p = t_1$. Again, we find the step $\kappa =: \kappa_2 > \kappa_1$ and the position $t =: t_2 < t_1$, so that for every $k \ge \kappa_2$

$$\sum_{i=t_2}^{M} c_i^{(k)} \quad \ge \quad 1 + \sum_{i=t_1}^{M} c_i^{(k)} \,.$$

In the same way we apply (3.3) and find steps $\kappa_3 < \kappa_4 < \kappa_5 < \dots$ and positions $t_3 > t_4 > t_5 \dots$ such that the digit sum $\sum_{i=t_s}^{M} c_i^{(\kappa_s)}$ increases with s at least by 1. Since there are infinitely many steps, the digit sum increases with s to infinity, which contradicts the fact that we started with the digit sum $\sum_{k=0}^{n}(a_k + b_k)$ and the transcription we use do not increase the digit sum. $\square$

In order to check whether β satisfies Property T we have to know all the minimal forbidden strings. Let the Rényi development of 1 be finite, i.e. $d_\beta(1) = t_1 t_2 \dots t_m$, then we have following candidates for minimal forbidden strings:

- The only minimal forbidden word of the length 1 is $t_1 + 1$: every smaller string is not strictly lexicographically greater than $d_\beta(1)$, all greater words do not fit the second condition in the Definition 3.1.1.
- The only minimal forbidden string of the length 2 is $t_1(t_2 + 1)$: the first digit has to be t_1 – every smaller digit causes the word to be admissible, all greater digits contradict the first condition in the definition of minimal forbidden string. If the second digit was smaller the whole word would been admissible, on the contrary if the second digit was greater the word would not fit the second condition in the Definition 3.1.1.
- Using the same argumentation, we can show that there is at most one minimal forbidden string for each length $k = 1, 2, \dots, n-1$: $t_1 t_2 \dots t_{k-1}(t_k + 1)$.
- The minimal forbidden string of the length n is $t_1 t_2 \dots t_n$.

Note that not all the above forbidden strings must be minimal. For example if β has the Rényi development of 1 being $d_\beta(1) = 111$, the above list of strings is equal to 2, 12, 111. However, 12 is not minimal.

In [6] it is shown that if β has a finite development of 1 with decreasing digits, then $\mathrm{Fin}(\beta)$ is closed under addition. The proof includes an algorithm for addition. Let us show that the result of [6] is a consequence of our Theorem 3.1.3.

Corollary 3.1.4. *Let* $d_\beta(1) = t_1 \dots t_m$, $t_1 \geq t_2 \geq \dots \geq t_m \geq 1$. *Then* $\mathrm{Fin}(\beta)$ *is closed under addition of positive elements.*

Proof. We shall verify the assumptions of Theorem 3.1.3. Consider the forbidden string $t_1 t_2 \dots t_{i-1}(t_i + 1)$, for $1 \leq i \leq m-1$. Clearly, if we consider the forbidden string the β-representation of a number x

$$x = t_1\beta^{i-1} + \dots + (t_{i-2})\beta^2 + t_{i-1}\beta + (t_i + 1)$$

by digit-wise summation of x with following two "zeros"[1]

$$\begin{gathered} \beta^i - t_1\beta^{i-1} - \dots - t_m\beta^{-m+i} \\ -1 + t_1\beta^{-1} + \dots + t_m\beta^{-m} \end{gathered}$$

we obtain following equality

$$\begin{aligned} &t_1\beta^{i-1} + \dots + t_{i-1}\beta + (t_i + 1) = \\ &= \beta^i + (t_1 - t_{i+1})\beta^{-1} + \dots + (t_{m-i} - t_m)\beta^{-(m-i)} + t_{m-i+1}\beta^{-(m-i+1)} + \dots + t_m\beta^{-m}. \end{aligned}$$

[1] i.e. β-representations of the number 0.

The assumption of the corollary assure that the coefficients on the right hand side are non-negative. The digit sum on the left and on the right is the same. Thus

$$1\underbrace{0\,0\ldots0}_{i\ \text{times}}(t_1-t_{i+1})(t_2-t_{i+2})\ldots(t_{m-i}-t_m)t_{m-i+1}\ldots t_m$$

is the desired finite string lexicographically strictly greater than $0\,t_1t_2\ldots t_{i-1}(t_i+1)$.

It remains to transcribe the string $t_1t_2\ldots t_{m-1}t_m$ into the lexicographically greater string $1\underbrace{0\,0...0}_{m\ \text{times}}$. □

The conditions of Theorem 3.1.3 are however satisfied also for other irrationals that do not fulfil assumptions of Corollary 3.1.4. As an example we may consider the minimal Pisot number. It is known that the smallest among all Pisot numbers is β solution of the equation $x^3 = x + 1$. The Rényi development of 1 is $d_\beta(1) = 10001$. The number β thus satisfies relations

$$\beta^3 = \beta + 1 \qquad \text{and} \qquad \beta^5 = \beta^4 + 1\,.$$

The minimal forbidden strings are 2, 11, 101, 1001, and 10001. Their transcription according to Property T is the following:

$$\begin{array}{rcl} 2 &=& \beta^2 + \beta^{-5} \\ \beta + 1 &=& \beta^3 \\ \beta^2 + 1 &=& \beta^3 + \beta^{-3} \\ \beta^3 + 1 &=& \beta^4 + \beta^{-5} \\ \beta^4 + 1 &=& \beta^5 \end{array}$$

The digit sum in every transcription is smaller or equal to the digit sum of the corresponding minimal forbidden string. Therefore $\mathrm{Fin}(\beta)$ is according to Theorem 3.1.3 closed under addition of positive numbers. Since $d_\beta(1)$ is finite, by Proposition 3.1.1 $\mathrm{Fin}(\beta)$ is a ring. This was shown already in [2].

In the assumptions of Theorem 3.1.3 the condition of non-increasing digit sum can be replaced by another requirement.We state it in the following theorem. Its proof uses the idea and notation of the proof of Theorem 3.1.3.

Theorem 3.1.5. *Let $\beta > 1$ be an algebraic integer satisfying Property T, and let at least one of its conjugates, say β', belong to $(0, 1)$. Then $\mathrm{Fin}(\beta)$ is closed under addition of positive elements. Moreover, for every positive $x, y \in \mathrm{Fin}(\beta)$, the β-expansion of $x+y$ can be obtained from any β-representation of $x + y$ using finitely many transcriptions.*

Proof. If it was possible to apply a transcription on the β-representation of $x + y$ infinitely many times, then we obtain the sequence of β-representations

$$x + y = \sum_{i=\ell_k}^{M} c_i^{(k)} \beta^i\,,$$

where the smallest indices of the non-zero digits ℓ_k satisfy $\lim_{k\to\infty} \ell_k = -\infty$. Here we have used the notation of the proof of Theorem 3.1.3. Now we use the isomorphism between algebraic fields $\mathbb{Q}[\beta]$ and $\mathbb{Q}[\beta']$ to obtain

$$(x+y)' = x' + y' = \sum_{i=\ell_k}^{M} c_i^{(k)} (\beta')^i \geq (\beta')^{\ell_k}\,.$$

The last inequality follows from the fact that $\beta' > 0$ and $c_i^{(k)} \geq 0$ for all k and i. Since $\beta' < 1$ we have $\lim_{k\to\infty} (\beta')^{\ell_k} = +\infty$, which is a contradiction. □

Remark. Let us point out that an algebraic integer β with at least one conjugate in the interval $(0,1)$ must have an infinite Rényi development of 1 (see remarks to Theorem 4.1.1). Such β has necessarily infinitely many minimal forbidden strings. The only examples known to the author of β satisfying Property T and having a conjugate $\beta' \in (0,1)$ have been treated in [6], namely those which have eventually periodic $d_\beta(1)$ with period of length 1,

$$d_\beta(1) = t_1 t_2 \dots t_{m-1} t_m t_m t_m \dots\,, \qquad \text{with } t_1 \geq t_2 \geq \cdots \geq t_m \geq 1\,. \tag{3.4}$$

In such a case every minimal forbidden string has a transcription with digit sum strictly smaller than its own digit sum. Thus closure of $\mathrm{Fin}(\beta)$ under addition of positive elements follows already by Theorem 3.1.3. This means that we don't know any β for which Theorem 3.1.5 would be necessary.

From the above remark one could expect that closure of $\mathrm{Fin}(\beta)$ under addition forces that digit sum of the transcriptions of minimal forbidden strings is smaller or equal than the digit sum of the corresponding forbidden string. It is not so. For example let β be the solution of $x^3 = 2x^2 + 1$. Then $d_\beta(1) = 201$ and the minimal forbidden string 3 has the β-expansion

$$3 = \beta + \frac{1}{\beta} + \frac{1}{\beta^2} + \frac{1}{\beta^3} + \frac{1}{\beta^4}\,.$$

The digit sum of this transcription of 3 is equal to 5. If there exists another transcription of 3 with digit sum ≤ 3, it must be lexicographically strictly larger than 03 and strictly smaller than 101111, because β-expansion is lexicographically greatest among all representations of a number. It can be shown easily that a string with the above properties does not exist.

In the same time $\mathrm{Fin}(\beta)$ is closed under addition. This follows from the results of Akiyama who shows that for a cubic Pisot unit β the set $\mathrm{Fin}(\beta)$ is a ring if and only if $d_\beta(1)$ is finite, see [2].

On the other hand, Property T is not sufficient for $\mathrm{Fin}(\beta)$ to be closed under addition of positive elements. As an example we can mention β with the Rényi development of 1 being $d_\beta(1) = 100001$. Such β satisfies $\beta^6 = \beta^5 + 1$. Among the conjugates of β there is a pair of complex conjugates, say β', $\beta'' = \overline{\beta'}$, with absolute value $|\beta'| = |\beta''| \doteq 1.0328$. Thus β is not a Pisot number and according to the result of [6], Fin(β) cannot be closed under addition of positive elements. However, Property T is satisfied for β. All minimal forbidden string can be transcribed as follows:

$$\begin{aligned}
2 &= \beta + \beta^{-6} + \beta^{-7} + \beta^{-8} + \beta^{-9} + \beta^{-10} \\
\beta + 1 &= \beta^2 + \beta^{-6} + \beta^{-7} + \beta^{-8} + \beta^{-9} \\
\beta^2 + 1 &= \beta^3 + \beta^{-6} + \beta^{-7} + \beta^{-8} \\
\beta^3 + 1 &= \beta^4 + \beta^{-6} + \beta^{-7} \\
\beta^4 + 1 &= \beta^5 + \beta^{-6} \\
\beta^5 + 1 &= \beta^6
\end{aligned}$$

Note that the transcriptions are finite, lexicographically strictly greater.

3.2 Program BetaArithmetic

As a part if this diploma thesis program performing arithmetic operations in some Pisot numeration systems was written. Actually there are three versions of the program – the workstation version for Win32 system, command line version performing batch tasks and also CGI version prepared for running on a web server.

All these three version differ one each other only by the (graphical) user interface, the underlaying arithmetic library is always the same. Therefore to avoid any troublesome changes of the code when transferring the library from one platform to another, the program was written in the object-oriented language C++ – standardized since 1998[2] and nowadays the most wide spread general-purpose programming language with compilers available for any platform.

Remark (C++ language tools). The fundamental principle of the object oriented programming – encapsulation of the data – was very suitable to this special arithmetic library as well as the possibility of overloading of the operators, which enables very intuitive use of the finished library. Finally the existence of standard template library (especially the template for a dynamical bidirectional list) made the creation of this library much more easier.

3.2.1 Class `CBetaExp`

Any number in a β-expansion is represented as an instance of the class `CBetaExp`. This class contains all the data items necessary for description of the expanded number. There are also defined member functions which perform manipulations with data – mainly overloaded versions of arithmetic operators.

Here is the complete declaration of the class CBetaExp:

```
class CBetaExp
{
  private:
    static const int maxExpLen = 100;
```

[2]ISO ITTF published the standard ISO/IEC 14882:1998 on September 1, 1998

```
    list<int> exp;
    int length;
    int sgn;

    void step1();
    void arrange1 (list<int>::iterator __i);
    int test1 (list<int>::iterator __i);

    int step2();
    void arrange2 (list<int>::iterator __i,int eqLen);
    int test2 (list<int>::iterator __i);

  public:
    static int lengthRenyi;
    static int *intRenyi;

    int getFpLen();
    string getExp();

    CBetaExp();
    CBetaExp(const CBetaExp& R);
    CBetaExp(int elementsNumber);
    CBetaExp(string expStr);

    ~CBetaExp();

    CBetaExp operator+ (CBetaExp& R);
    CBetaExp operator- (CBetaExp& R);
    CBetaExp operator* (CBetaExp& R);

    friend ostream& operator<< (ostream& Str, CBetaExp& R);
    friend istream& operator>> (istream& Str, CBetaExp& R);
};
```

A single digit of an expansion is stored as a member of the dynamical bidirectional list of integers named `exp`; the template

```
template <class T, class Allocator = allocator<T> > class list;
```

from the Standard Template Library was used for this purpose. Although the use of this dynamical structure has an impact on the speed of the computation, it eliminates the need of limits on the length of either the fractional part or the whole expansion.

The other attributes store the overall length of the expansion (`int length`) and the sign of the expansion (`int sgn`). Furthermore we have to represent the numeration system somehow, as it is necessary for performing the arithmetic operations correctly. There are two public static attributes for this purpose – `int lengthRenyi` stores the length (the number of digits) of the Rényi development of unit, whereas the integer array `int *intRenyi` contains its actual digits.

The member functions can be imaginarily divided into several categories: two groups of private functions for performing single steps of the algorithm for addition, auxiliary functions accessing the attributes, the constructors and the destructor and overloaded arithmetic operators. There are also defined two friendly functions – overloaded operators `operator>>` and `operator<<` – to enable easier input and output of the expansion.

X	Y		X + Y	X − Y
<0	<0	$\lvert X\rvert > \lvert Y\rvert$	$-(\lvert X\rvert + \lvert Y\rvert)$	$-(\lvert X\rvert - \lvert Y\rvert)$
<0	<0	$\lvert X\rvert > \lvert Y\rvert$	$-(\lvert X\rvert + \lvert Y\rvert)$	$\lvert Y\rvert - \lvert X\rvert$
<0	>0	$\lvert X\rvert > \lvert Y\rvert$	$-(\lvert X\rvert - Y)$	$-(\lvert X\rvert + Y)$
<0	>0	$\lvert X\rvert > \lvert Y\rvert$	$Y - \lvert X\rvert$	$-(\lvert X\rvert + Y)$
>0	<0	$\lvert X\rvert > \lvert Y\rvert$	$X - \lvert Y\rvert$	$X + \lvert Y\rvert$
>0	<0	$\lvert X\rvert > \lvert Y\rvert$	$-(\lvert Y\rvert - X)$	$X + \lvert Y\rvert$
>0	>0	$\lvert X\rvert > \lvert Y\rvert$	$X + Y$	$X - Y$
>0	>0	$\lvert X\rvert > \lvert Y\rvert$	$X + Y$	$-(Y - X)$

Table 3.1: Addition and subtraction of two numbers X and Y in β-expansion

All the arithmetic operators does not change any of their parameters, they just create the copy of the expansion represented by their class instance (using the copy constructor `CBetaExp(const CBetaExp& R)`) and then add to it (subtract from or multiply by) the number represented by their parameter.

All the operators take care of the sign and absolute value of their parameters (however this is very simple in case of the multiplication) and if necessary they change addition to subtraction or vice versa (see Table 3.1).

3.2.2 Operator `CBetaExp::operator+ (CBetaExp&)`

The algorithm of addition can be divided into three parts

1. The inspection of the signs of the expansions and possible change of the addition to the subtraction or vice versa, according to the Table 3.1.
2. Stepwise addition of respective digits of the expansions, the result is a representation of the sum of expansions.
3. The normalization of the representation (i.e. the transformation to the expansion).

The first step is demonstrated in the Table 3.1, the second step is very easy to imagine as well as to perform, whereas the third part is the most important one. It is performed almost exactly as described in the Proposition 3.1.2. Here is respective piece of the code of the function

```
CBetaExp CBetaExp::operator+ (CBetaExp& R) {

  ...

    // copy of this expansion to store the result
  CBetaExp newR(*this);

    // the difference between the greatest powers in the expnasions
  int maxPowerDiff = newR.length - 1 - ( R.length - 1 );

    // if newR has smaller power by the greatest non-zero coefficient
    // refill it with zeros
  if ( maxPowerDiff < 0 ) {
    newR.exp.insert(newR.exp.begin(),abs(maxPowerDiff),0);
    newR.length += abs(maxPowerDiff);
  }

    // refill with zeros also the tail of newR if it is shorter
  if ( (newR.exp.size()-newR.length) < (R.exp.size()-R.length) )
    newR.exp.insert(newR.exp.end(),
                  R.exp.size()-R.length-newR.exp.size()+newR.length,0);

  list<int>::const_iterator iR = R.exp.begin();
  list<int>::iterator inewR = newR.exp.begin();

    // set up the iterators to the same powers
  if ( maxPowerDiff > 0 )
    while (maxPowerDiff > 0) {
    inewR++;
    maxPowerDiff--;
  }

    // the digitwise addition
  while ( iR != R.exp.end() ) {
    *inewR += *iR;
    inewR++;
    iR++;
  }

    // the normalization
  int chngCtrl;
  do {
    newR.step1();
    chngCtrl = newR.step2();
  } while (chngCtrl != 0 );
```

```
  return newR;
}
```

The normalization is performed in the last `do` loop. The function `step1()` firstly replaces all the occurrences of the Rényi development of unit by relevant power of β, afterwards the function `step2()` finds the most left occurrence of a minimal forbidden string and replaces it with the respective string according the Proposition 3.1.2. The function returns zero if no replacement was done and therefore the representation has already been transformed into the admissible β-expansion of the result.

3.3 The generation of the β-integers

In order to find bottom bounds on the constants $L_\oplus$ and $L_\otimes$ we have to let the program perform respective operation on the appropriately large set of different numbers (β-expansions).

The most suitable method is to pre-generate the numbers (for example all β-expansions of the length between 1 and 20) and afterwards let the program add/multiply everyone with/by each other. Therefore the problem of finding good lower bounds is reduced to the generation of the sufficiently large set of β-expansions (and obviously to the problem of required computational time).

We show the method how to cope with this task in the case β is n-generalized golden mean. The general result is formulated at the end of this section.

Let β be the base of the numeration system of n-generalized golden mean, i.e. β is the root of the equation

$$x^n = x^{n-1} + x^{n-2} + \ldots + x + 1$$

hence

$$\beta^n = \beta^{n-1} + \beta^{n-2} + \ldots + \beta + 1 \tag{3.5}$$

or

$$\beta^{n+1} = \beta^n + \beta^{n-1} + \ldots + \beta^2 + \beta\,. \tag{3.6}$$

If we use (3.5) instead of β^n in (3.6), we get

$$\beta^{n+1} = 2\beta^{n-1} + 2\beta^{n-1} + 2\beta + 1\,.$$

Using the same technique we can express an arbitrary power of β in the form

$$\beta^k = F_k^{(n-1)}\beta^{n-1} + F_k^{(n-2)}\beta^{n-2} + \ldots + F_k^{(1)}\beta + F_k^{(0)} \tag{3.7}$$

where $(F_k^{(n-1)})_{k=0}^\infty, \ldots, (F_k^{(0)})_{k=0}^\infty$ are sequences of non-negative integers.

By multiplying the equation (3.7) with the number β or by shifting the coefficients we obtain equations

$$\beta^{k+1} = F_k^{(n-1)}(\underbrace{\beta^{n-1} + \beta^{n-2} + \ldots + \beta + 1}_{\beta^n}) + F_k^{(n-2)}\beta^{n-1} + \ldots + F_k^{(0)}\beta \tag{3.8a}$$

or

$$\beta^{k+1} = F_{k+1}^{(n-1)}\beta^{n-1} + F_{k+1}^{(n-2)}\beta^{n-2} + \ldots + F_{k+1}^{(1)}\beta + F_{k+1}^{(0)} \tag{3.8b}$$

which will help us in search for the recurrent formula for elements of the sequence $(F_k^{(n-1)})_{k=0}^{\infty}$.

Comparing the coefficients in (3.8) at the same powers of β we obtain following equations

$$\begin{aligned}
F_{k+1}^{(n-1)} &= F_k^{(n-1)} + F_k^{(n-2)} \\
F_{k+1}^{(n-2)} &= F_k^{(n-1)} + F_k^{(n-3)} \\
&\vdots \\
F_{k+1}^{(1)} &= F_k^{(n-1)} + F_k^{(0)} \\
F_{k+1}^{(0)} &= F_k^{(n-1)}
\end{aligned}$$

by consecutive substitutions from bottom to above, we get sought recurrent formula

$$F_{k+1}^{(n-1)} = F_k^{(n-1)} + F_{k-1}^{(n-1)} + \ldots + F_{k-n+1}^{(n-1)}$$

where $F_0^{(n-1)} = F_1^{(n-1)} = \ldots = F_{n-2}^{(n-1)} = 0$ and $F_{n-1}^{(n-1)} = 1$.

Definition 3.3.1. *The sequence $(F_k)_{k=0}^{\infty}$ of non-negative integers, such that $F_0 = F_1 = \ldots = F_{n-2} = 0$ and $F_{n-1} = 1$ and for $\forall k \in \mathbb{N}, k \geq n-1$*

$$F_{k+1} = F_k + F_{k-1} + \ldots + F_{k-n+1}\,, \tag{3.9}$$

is called called n-generalized Fibonacci sequence.

Proposition 3.3.1. *For any positive integer x there exist $N \in \mathbb{Z}, N \leq n$ and non-negative coefficients $\alpha_N, \alpha_{N-1}, \ldots, \alpha_n \in \mathbb{Z}$ such that*

$$x = \sum_{k=n}^{N} \alpha_k F_k \tag{3.10}$$

where F_k are the elements of n-generalized Fibonacci sequence.

Proof. We describe the algorithm, which finds this form for a given $x \in \mathbb{N}$: at first we find the greatest $l_1 \in \mathbb{N}$, such that $F_{l_1} \leq x$, we put $\alpha_{l_1} = 1$ and then we continue in the same way for the rest of the number – we search for the greatest $l_2 \in \mathbb{N}$, such that $F_{l_2} \leq (x - F_{l_1})$, we put $\alpha_{l_2} = 1$ and so on, until we express the whole number x. □

Coefficients obtained by this algorithm will be called the representation of the number x in n-generalized Fibonacci sequence; we denote it

$$(x)_F := (\alpha_l \alpha_{l-1} \ldots \alpha_n)$$

Proposition 3.3.2. *Let $(\alpha_l \alpha_{l-1} \ldots \alpha_n)$ be a representation of an natural number x x in n-generalized Fibonacci sequence. Then $\alpha_l \alpha_{l-1} \ldots \alpha_n \bullet$ is an admissible β-expansion in the numeration system of corresponding n-generalized golden mean.*

Proof. 1. In the representation $(x)_F$ of any x there cannot emerge the forbidden factor $(1)^n$, as it is equal to the factor $1(0)^n$ (vide (3.9)), which starts one position higher. This implies there will be the factor $1(0)^n$ on the output of the "greedy algorithm" proceeding from the top to the bottom.

2. In similar way (using the equation (3.9)) we can show, that there will be no 2 (or even greater coefficient) in the representation $(x)_F$ of arbitrary x:

$$\begin{aligned} 2F_k &= F_k + F_{k-1} + F_{k-2} + \ldots + F_{k-n} \\ &= F_{k+1} + F_{k-n} \end{aligned}$$

hence

$$2F_k \geq F_{k+1}$$

and the algorithm again selects the expression starting on the higher position.

□

Corollary 3.3.3. *Let $p \in \mathbb{N}$. The representations of first p natural numbers in a n-generalized Fibonacci sequence, i.e. the words $(1)_F, (2)_F, \ldots, (p)_F$, coincide with the β-expansions of first p numbers from the set $\mathbb{Z}_\beta$.*

Consider β different from n-generalized golden mean, we can use the results from [4].

Let β be a Pisot number with finite Rényi development of unit $d_\beta(1) = t_1 \ldots t_m$. Then

$$P(X) = X^m - t_1 X^{m-1} - \ldots - t_m$$

is the *characteristic polynomial of* β. To P is associated a linear recurrent sequence of integers U_β, defined by

$$\begin{gathered} u_{n+m} = t_1 u_{n+m-1} + \ldots + t_m u_n \\ u_0 = 1, \quad u_i = t_1 u_{i-1} + \ldots + t_i u_0 + 1, \quad 1 \leq i \leq m-1. \end{gathered}$$

Note that the characteristic polynomial is a multiple in $\mathbb{Z}[X]$ of the minimal polynomial of β.

The increasing sequence U_β defines the *numeration system associated with* β. In a correspondence with the case of a representation in a n-generalized Fibonacci sequence, we will denote the greedy U_β-representation of a positive integer x by the symbol $(x)_{U_\beta}$.

We then have the following result [4].

Proposition 3.3.4. *Let $p \in \mathbb{N}$. The greedy U_β-representations of first p natural numbers, i.e. the words $(1)_{U_\beta}, (2)_{U_\beta}, \ldots, (p)_{U_\beta}$, coincide with the β-expansions of first p numbers from the set $\mathbb{Z}_\beta$.*

3.4 Lower bounds on $L_\oplus$ and $L_\otimes$

In order to determine the lower bounds on $L_\oplus$ and $L_\otimes$ we have used computer program BetaArithmetic to perform additions and multiplications on a large set of β-expansions generated by menas of the method described in the last section.

All the pre-generated β-expansions as well as the results of computations are stored on the companion CD-ROM (see Appendix C).

3.4.1 Tribonacci number, β solution of $x^3 = x^2 + x + 1$

As a result for Tribonacci number numeration system, we obtained examples of a sum with 5 and product with 4 fractional digits, namely:

$$1001011010 + 1001011011 = 10100100100 \bullet 10101$$
$$110100100101101 \times 110100100101101 = 1100100010001000010010010111011 \bullet 0011$$

These were the first obtained while performing the addition or multiplication on all β-integers progressively from $1 + 1 = 10.001$ or $1 \times 1 = 1$. Hence it was necessary to perform 251 001 additions and 112 487 236 multiplications to find these examples.

3.4.2 Minimal Pisot number, β solution of $x^3 = x + 1$

As a result for minimal Pisot number numeration system, we obtained examples of a sum with 13 and product with 8 fractional digits, namely:

$$10000100001 + 10000000100001 = 100001000000010 \bullet 0001000000001$$
$$100001 \times 100001 = 100000010000 \bullet 01000001$$

This time 2 607 additions and 49 multiplications were enough to find these examples. This lower bound on the coefficient $L_\oplus$ will enable us to find exact value of this coefficient. On the other hand for the coefficient $L_\otimes$ there is still quite large interval of possible values. Unfortunately we were not able find an example of multiplication giving higher number of fractional digits, however hard we had tried.

3.4.3 Pisot number β solution of $x^3 = 25x^2 + 15x + 2$

In this numeration system the obtained lower bounds are 5 for addition and 7 for multiplication. The found examples are very simple:

$$(25)(0)(25) + (25)(0)(25) = (1)(24)(12)(11) \bullet (23)(0)(14)(13)(2)$$
$$(25) \times (25) = (24)(10) \bullet (21)(24)(16)(7)(16)(13)(2)$$

Chapter 4

Upper bounds on $L_\oplus$ and $L_\otimes$

In this chapter we explain two methods for determining upper bounds on the number of fraction digits that arise under addition and multiplication of β-integers.

In the first section we show the H-K method, which stems from the theorem that we cite from [7]. In following two sections we use this method to find the upper bounds in the Tribonacci numeration system and Minimal pisot numeration system.

Unfortunately there are algebraic numbers β, such that the first method cannot be applied in associated numeration systems. We explain second method, which can be used in some of these "bad" cases. Nevertheless also this method has a limitation in use – it is applicable only to β being a Pisot number.

In the last section there is the demonstration of use of the second method for 25-15-2 Pisot number.

4.1 One-conjugate method

Theorem 4.1.1. *Let β be an algebraic number, $\beta > 1$, with at least one conjugate β' satisfying*

$$\begin{aligned} H &:= \sup\{|z'| \mid z \in \mathbb{Z}_\beta\} < +\infty \\ K &:= \inf\{|z'| \mid z \in \mathbb{Z}_\beta \setminus \beta\mathbb{Z}_\beta\} > 0 \end{aligned}$$

Then

$$\left(\frac{1}{|\beta'|}\right)^{L_\oplus} < \frac{2H}{K} \qquad \text{and} \qquad \left(\frac{1}{|\beta'|}\right)^{L_\otimes} < \frac{H^2}{K}. \tag{4.1}$$

Remark.

- Since $H \geq \sup\{|\beta'^k| \mid k \in \mathbb{N}\}$, the condition $H < +\infty$ implies that $|\beta'| < 1$. In this case
$$H \leq \sum_{i=0}^{\infty} \lfloor \beta \rfloor |\beta'|^i = \frac{\lfloor \beta \rfloor}{1 - |\beta'|}.$$
- If $\beta' \in (0,1)$, then $K = 1$. Obviously, for $z = \sum_{i=0}^{n} z_i \beta^i$, $z_0 \neq 0$ we have
$$z' = \sum_{i=0}^{n} z_i (\beta')^i \geq z_0 \geq 1.$$

- If $d_\beta(1)$ is finite, then $\beta' \notin (0,1)$. Let $\beta > 1$ be an algebraic number with finite Rényi development of unit
$$d_\beta(1) = t_1 t_2 \dots t_m$$
such that $t_m \neq 0$. Then
$$1 = \frac{t_1}{\beta} + \frac{t_2}{\beta^2} + \dots + \frac{t_m}{\beta^m}\,. \tag{4.2}$$
Suppose by a contradiction that $\beta' \in (0,1)$. Then $\beta' > (\beta')^2 > \dots > (\beta')^m$. Hence conjugating the equation (4.2) we get
$$1 = \frac{t_1}{\beta'} + \frac{t_2}{(\beta')^2} + \ldots + \frac{t_m}{(\beta')^m} > \frac{t_m}{(\beta')^m} \geq \frac{1}{(\beta')^m} \tag{4.3}$$
which implies $(\beta')^m > 1$. Hence $\beta' > 1$, which is a contradiction.

If the considered algebraic conjugate β' of β is negative or complex, it is complicated to determine the value of K and H. However, for obtaining bounds on $L_\oplus$, $L_\otimes$ it suffices to have a "reasonable" estimates on K and H. In order to determine good approximation of K and H we introduce some notation. For $n \in \mathbb{N}$ we shall consider the set
$$E_n := \{z \in \mathbb{Z}_\beta \mid 0 \leq z < \beta^n\}\,.$$
In fact this is the set of all $a_0 + a_1\beta + \dots + a_{n-1}\beta^{n-1}$ where $a_{n-1}\dots a_1 a_0$ is an admissible β-expansion. We denote
$$\begin{aligned} \min\nolimits_n &:= \min\{|z'| \mid z \in E_n, z \notin \beta\mathbb{Z}_\beta\}\,, \\ \max\nolimits_n &:= \max\{|z'| \mid z \in E_n\}\,. \end{aligned}$$

Lemma 4.1.2. *Let $\beta > 1$ be an algebraic number with at least one conjugate β' in modulus less than 1. Then*

(i) For all $n \in \mathbb{N}$ we have $K \geq K_n := \min_n - |\beta'|^n H$.

(ii) $K > 0$ if and only if there exists $n \in \mathbb{N}$ such that $K_n > 0$.

Proof. (i) Let $z \in \mathbb{Z}_\beta \setminus \beta\mathbb{Z}_\beta$. Then $z = \sum_{i=0}^N b_i\beta^i$, $b_1 \neq 0$. The triangle inequality gives
$$|z'| \geq \left|\sum_{i=0}^{n-1} b_i {\beta'}^i\right| - \left|\sum_{i=n}^{N} b_i {\beta'}^i\right| \geq \min\nolimits_n - |\beta'|^n \left|\sum_{i=n}^{N} b_i {\beta'}^{i-n}\right| > \min\nolimits_n - |\beta'|^n H = K_n\,.$$
Hence taking the infimum on both sides we obtain $K \geq K_n$.

(ii) From the definition of $\min_n$ it follows that $\min_n$ is a decreasing sequence with $\lim_{n\to\infty} \min_n = K$. If there exists $n \in \mathbb{N}$ such that $\min_n - |\beta'|^n H > 0$ we have $K > 0$ from (i). The opposite implication follows easily from the fact that $\lim_{n\to\infty} K_n = K$. ∎

For a fixed β, the determination of $\min_n$ for small n is relatively easy. It suffices to find the minimum of a finite set with small number of elements. If for such n we have $K_n = \min_n - |\beta'|^n H > 0$, we obtain using (4.1) bounds on $L_\oplus$ and $L_\otimes$. We illustrate this procedure on β – the Tribonacci number.

4.2 $L_\oplus$, $L_\otimes$ for the Tribonacci number

Let β be the real root of $x^3 = x^2 + x + 1$. The arithmetics on β-expansions was already studied in [9]. Messaoudi [9] finds the upper bound on the number of β-fractional digits for the Tribonacci multiplication as 9. Arnoux, see [9], conjectures that $L_\otimes = 3$. We refute the conjecture of Arnoux and find a better bound on $L_\otimes$ than 9. Moreover, we find the bound for $L_\oplus$, as well.

It turns out that the best estimates on $L_\oplus$, $L_\otimes$ are obtained by Theorem 4.1.1 with approximation of K by K_n for $n = 9$. By inspection of the set E_9 we obtain

$$\min\nolimits_9 = |1 + {\beta'}^2 + {\beta'}^4 + {\beta'}^7| \doteq 0.5465$$

and

$$\max\nolimits_9 = |1 + {\beta'}^3 + {\beta'}^6| \doteq 1.5444\,.$$

Consider $y \in \mathbb{Z}_\beta$, $y = \sum_{k=0}^{N} a_k \beta^k$. Then from the triangle inequality

$$\begin{aligned} |y'| &\le \left|\sum_{k=0}^{8} a_k {\beta'}^k\right| + |\beta'|^9 \left|\sum_{k=9}^{17} a_k {\beta'}^{k-9}\right| + |\beta'|^{18} \left|\sum_{k=18}^{26} a_k {\beta'}^{k-18}\right| + \cdots \\ &< \max\nolimits_9 \left(1 + |\beta'|^9 + |\beta'|^{18} + \cdots\right) = \frac{\max_9}{1 - |\beta'|^9}\,. \end{aligned}$$

In this way we have obtained an upper estimate on H, i.e. $H \le \frac{\max_9}{1-|\beta'|^9}$. This implies

$$K_9 = \min\nolimits_9 - |\beta'|^9 H \ge \min\nolimits_9 - |\beta'|^9 \frac{\max_9}{1 - |\beta'|^9}\,.$$

Hence

$$\begin{aligned} \left(\frac{1}{|\beta'|}\right)^{L_\oplus} &< \frac{2H}{K} \le \; 2\frac{\max_9}{1-|\beta'|^9}\left(\min\nolimits_9 - |\beta'|^9 \frac{\max_9}{1-|\beta'|^9}\right)^{-1} \doteq 7.5003 \\ \left(\frac{1}{|\beta'|}\right)^{L_\otimes} &< \frac{H^2}{K} \le \; \left(\frac{\max_9}{1-|\beta'|^9}\right)^{2}\left(\min\nolimits_9 - |\beta'|^9 \frac{\max_9}{1-|\beta'|^9}\right)^{-1} \doteq 6.1908 \end{aligned}$$

Since

$$\left(\frac{1}{|\beta'|}\right)^5 \doteq 4.5880\,, \quad \left(\frac{1}{|\beta'|}\right)^6 \doteq 6.2222\,, \quad \left(\frac{1}{|\beta'|}\right)^7 \doteq 8.4386\,,$$

we conclude that $L_\oplus \le 6$, $L_\otimes \le 5$.

Considering the lower bounds on these coefficients for Tribonacci number, we obtained experimentally by the use of program BetaArithmetic (see Section 3.4, Appendix C), we can sum up our results in the following corollary.

Corollary 4.2.1. *Let β be the Tribonacci number, then*

$$5 \le L_\oplus(\beta) \le 6 \qquad \textit{and} \qquad 4 \le L_\otimes(\beta) \le 5\,.$$

4.3 $L_\oplus$, $L_\otimes$ for the minimal Pisot number

In the case of minimal Pisot number, finding suitable approximation of K took a little bit more steps and computation. The best estimates were found for $n = 37$ (for the coefficient $L_\oplus$) and $n = 35$ (for the coefficient $L_\otimes$).

By inspection of the set E_{35} or E_{37} we obtained

$$\begin{aligned} \min\nolimits_{35} &= |1+\beta^9+\beta^{14}+\beta^{22}+\beta^{27}+\beta^{32}| \doteq 0.5202 \\ \max\nolimits_{35} &= |1+\beta^5+\beta^{10}+\beta^{18}+\beta^{23}+\beta^{28}+\beta^{33}| \doteq 1.8135 \end{aligned}$$

or

$$\begin{aligned} \min\nolimits_{37} &= |1+\beta^9+\beta^{14}+\beta^{22}+\beta^{27}+\beta^{32}| \doteq 0.5202 \\ \max\nolimits_{37} &= |1+\beta^5+\beta^{10}+\beta^{18}+\beta^{23}+\beta^{28}+\beta^{36}| \doteq 1.8148 \end{aligned}$$

the whole convergency to these values is listed in Appendix A.

Using the same technique as in the last section, we obtain

$$\begin{aligned} \left(\frac{1}{|\beta'|}\right)^{L_\oplus} &= \frac{2H}{K} \doteq 7.1540 \\ \left(\frac{1}{|\beta'|}\right)^{L_\otimes} &= \frac{H^2}{K} \doteq 6.5843\,. \end{aligned}$$

Since

$$\left(\frac{1}{|\beta'|}\right)^{12} \doteq 5.4043 \qquad \left(\frac{1}{|\beta'|}\right)^{13} \doteq 6.2201 \qquad \left(\frac{1}{|\beta'|}\right)^{14} \doteq 7.1592$$

we conclude that $L_\oplus \le 13$, $L_\otimes \le 13$.

When we put these upper bounds together with the lower ones obtained by means of the programm BetaArithmetic, we have following corollary.

Corollary 4.3.1. *Let β be the minimal Pisot number, then*

$$L_\oplus(\beta) = 13 \qquad \textit{and} \qquad 8 \le L_\otimes(\beta) \le 13\,.$$

4.4 Case $K = 0$

The one-conjugate method mentioned in the Section 4.1 cannot be used in the case that $K = 0$. It is however difficult to prove $K = 0$ for a given algebraic β and its conjugate β'. Particular situation is solved by the following proposition.

Proposition 4.4.1. *Let $\beta > 1$ be an algebraic number and $\beta' \in (-1, 0)$ its conjugate such that $\frac{1}{\beta'^2} < \lfloor\beta\rfloor$. Then $K = 0$.*

Proof. Set $\gamma := \beta'^{-2}$. Digits in the γ-expansion take values in the set $\{0, 1, \ldots, \lfloor\gamma\rfloor\}$. Since $\lfloor\gamma\rfloor \le \lfloor\beta\rfloor - 1$ and the Rényi development of unit $d_\beta(1)$ is of the form

$$d_\beta(1) = \lfloor\beta\rfloor t_2 t_3 \ldots$$

every sequence of digits in $\{0,1,\ldots,\lfloor\gamma\rfloor\}$ is lexicographically smaller than $d_\beta(1)$ and thus is an admissible β-expansion.

Since $1 < -\beta'^{-1} < \gamma$, the γ-expansion of $-\beta'^{-1}$ has the form

$$-\beta'^{-1} = c_0 + c_1\gamma^{-1} + c_2\gamma^{-2} + c_3\gamma^{-3} + \cdots \tag{4.4}$$

where all coefficients $c_i \le \lfloor\beta\rfloor - 1$.

Let us define the sequence

$$z_n := 1 + c_0\beta + c_1\beta^3 + c_2\beta^5 + \cdots + c_n\beta^{2n+1}.$$

Clearly, $z_n \in \mathbb{Z}_\beta \setminus \beta\mathbb{Z}_\beta$ and

$$z_n' := 1 + \beta'(c_0 + c_1\beta'^2 + c_2\beta'^4 + \cdots + c_n\beta'^{2n}).$$

According to (4.4) we have $\lim_{n\to\infty} z_n' = 1 + \beta'(-\beta'^{-1}) = 0 = \lim_{n\to\infty} |z_n'|$. Finally, this implies $K = 0$. □

Example. *As an example of an algebraic number satisfying assumptions of Proposition 4.4.1 is $\beta > 1$ solution of the equation $x^3 = 25x^2 + 15x + 2$. The algebraic conjugates of $\beta \doteq 25.5892$ are $\beta' \doteq -0.38758$ and $\beta'' \doteq -0.20165$, and so $K = 0$ for both of them. Hence Theorem 4.1.1 cannot be used for determining the bounds on $L_\oplus$, $L_\otimes$. We thus present another method for finding these bounds and illustrate it further on the mentioned example.*

Note that similar situation happens infinitely many times, for example for a class of totally real cubic numbers, solutions to $x^3 = p^6x^2 + p^4x + p$, for $p \ge 3$. Theorem 4.1.1 cannot be applied to any of them which justifies utility of a new method.

4.5 Upper bounds on $L_\oplus$, $L_\otimes$ for β Pisot number

The second method of determining upper bounds on $L_\oplus$, $L_\otimes$ studied in this paper is applicable to β being a Pisot number. This method is based on the so-called cut-and-project scheme.

Let $\beta > 1$ be an algebraic integer of degree d, let $\beta^{(2)},\ldots,\beta^{(s)}$ be its real conjugates and let $\beta^{(s+1)}$, $\beta^{(s+2)} = \overline{\beta^{(s+1)}}$, $\ldots$, $\beta^{(d-1)}$, $\beta^{(d)} = \overline{\beta^{(d-1)}}$ be its non-real conjugates.

Then there exists a basis $\vec{y}^{(1)}$,$\vec{y}^{(2)}$, $\ldots$, $\vec{y}^{(d)}$ of the space $\mathbb{R}^d$ such that every $\vec{x} = (a_0, a_1, \ldots, a_{d-1}) \in \mathbb{Z}^d$ has in this basis the form

$$\vec{x} = \alpha_1\vec{y}^{(1)} + \alpha_2\vec{y}^{(2)} + \cdots + \alpha_d\vec{y}^{(d)},$$

where

$$\alpha_1 = a_0 + a_1\beta + a_2\beta^2 + \cdots + a_{d-1}\beta^{d-1} =: z \in \mathbb{Q}[\beta]$$

and

$$\begin{array}{ll} \alpha_i = z^{(i)} & \text{for } i = 2,3,\ldots,s, \\ \alpha_j = \Re(z^{(j)}) & \text{for } s < j \le d,\ j \text{ odd}, \\ \alpha_j = \Im(z^{(j)}) & \text{for } s < j \le d,\ j \text{ even}. \end{array} \tag{4.5}$$

The details of the construction of the basis $\vec{y}^{(1)},\vec{y}^{(2)}, \ldots, \vec{y}^{(d)}$ can be found in [1, 7]: At first we find (possibly) complex vectors

$$(x^{(1)})^T = (x_0^{(1)}, x_1^{(1)}, \ldots, x_{d-1}^{(1)}), \quad \ldots \quad , \quad (x^{(d)})^T = (x_0^{(d)}, x_1^{(d)}, \ldots, x_{d-1}^{(d)}),$$

such that for any $\vec{x} = (a_0, a_1, \ldots, a_{d-1}) \in \mathbb{Z}^d$ we have

$$\vec{x} = \alpha^{(1)}\vec{x}^{(1)} + \alpha^{(2)}\vec{x}^{(2)} + \cdots + \alpha^{(d)}\vec{x}^{(d)}\,, \tag{4.6}$$

where

$$\alpha^{(i)} = a_0 + a_1\beta^{(i)} + a_2(\beta^{(i)})^2 + \cdots + a_{d-1}(\beta^{(i)})^{d-1}, \quad \forall i \in \{1, \ldots, d\}.$$

If we denote by $\mathbb{X}$ the $d \times d$ matrix with $(\mathbb{X})_{ij} = x_j^{(i)}$ and if we expand the equation (4.6) into system of equations for coefficients $a_0, a_1, \ldots, a_{d-1}$

$$\begin{aligned}
a_0 = a_0 \left((\beta^{(1)})^0 x_0^{(1)} + \ldots + (\beta^{(d)})^0 x_0^{(d)}\right) + a_1 \left((\beta^{(1)})^1 x_0^{(1)} + \ldots + (\beta^{(d)})^1 x_0^{(d)}\right) + \ldots \\
\ldots + a_{d-1} \left((\beta^{(1)})^{d-1} x_0^{(1)} + \ldots + (\beta^{(d)})^{d-1} x_0^{(d)}\right) \\
\vdots \qquad\qquad \\
a_{d-1} = a_0 \left((\beta^{(1)})^0 x_{d-1}^{(1)} + \ldots + (\beta^{(d)})^0 x_{d-1}^{(d)}\right) + a_1 \left((\beta^{(1)})^1 x_{d-1}^{(1)} + \ldots + (\beta^{(d)})^1 x_{d-1}^{(d)}\right) + \ldots \\
\ldots + a_{d-1} \left((\beta^{(1)})^{d-1} x_{d-1}^{(1)} + \ldots + (\beta^{(d)})^{d-1} x_{d-1}^{(d)}\right)
\end{aligned}$$

it is easy to see that (4.6) holds for each $\vec{x}$ if and only if

$$I_d = \mathbb{V}(\beta^{(1)}, \ldots, \beta^{(d)}) \cdot \mathbb{X}$$

where $\mathbb{V}(\beta^{(1)}, \ldots, \beta^{(d)})$ is the Vandermonde matrix in variables $\beta^{(1)}, \ldots, \beta^{(d)}$,

$$\mathbb{V}(\beta^{(1)}, \ldots, \beta^{(d)}) := \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \beta^{(1)} & \beta^{(2)} & \cdots & \beta^{(d)} \\ \vdots & \vdots & \ddots & \vdots \\ (\beta^{(1)})^{d-1} & (\beta^{(2)})^{d-1} & \cdots & (\beta^{(d)})^{d-1} \end{pmatrix}.$$

The determinant of $\mathbb{V}(\beta^{(1)}, \ldots, \beta^{(d)})$ is equal to $\prod_{d\geq i>j\geq 1}(\beta^{(i)} - \beta^{(j)})$. Since all conjugates are distinct, the determinant is non-zero.

Using the Cramer rule to compute $x_j^{(i)}$, we obtain that $\vec{x}^{(i)}$ is real if $\beta^{(i)}$ is real, and if $\beta^{(j)}$ and $\beta^{(j+1)}$ are mutually complex conjugated roots, then $\vec{x}^{(j)} = \overline{\vec{x}^{(j+1)}}$.

Thus we can define a real basis $\vec{y}^{(1)}, \ldots, \vec{y}^{(d)}$ of $\mathbb{R}^d$ in a such way that

$$\begin{aligned}
&\vec{y}^{(i)} = \vec{x}^{(i)} && \text{if } \vec{x}^{(i)} \text{ is a real vector} \\
&\left.\begin{aligned} \vec{y}^{(j)} &= \vec{x}^{(j)} + \overline{\vec{x}^{(j)}} \\ \vec{y}^{(j+1)} &= i(\vec{x}^{(j)} - \overline{\vec{x}^{(j)}}) \end{aligned}\right\} && \text{if } \vec{x}^{(j)} \text{ and } \vec{x}^{(j+1)} \text{ are mutually complex conjugated}
\end{aligned}$$

Note that really the coordinates of a vector $\vec{x} = (a_0, a_1, \ldots, a_{d-1}) \in \mathbb{Z}^d$ with a respect to the basis $\vec{y}^{(1)}$,$\vec{y}^{(2)}$, $\ldots$, $\vec{y}^{(d)}$ are these written out in (4.5).

Let us denote

$$\mathbb{Z}[\beta] := \{a_0 + a_1\beta + a_2\beta^2 + \cdots + a_{d-1}\beta^{d-1} \mid a_i \in \mathbb{Z}\}\,.$$

i.e. the ring of polynomials in β with integer coefficients.

For β an algebraic integer, the set $\mathbb{Z}[\beta]$ is a ring, moreover if we put $V_1 = \mathbb{R}\vec{y}^{(1)}$ and $V_2 = \mathbb{R}\vec{y}^{(2)} + \mathbb{R}\vec{y}^{(3)} + \cdots + \mathbb{R}\vec{y}^{(d)}$, the set $\mathbb{Z}[\beta]$ is the projection of the lattice $\mathbb{Z}^d$ on V_1 along V_2. The correspondence

$$(a_0, a_1, \ldots, a_{d-1}) \mapsto a_0 + a_1\beta + \cdots + a_{d-1}\beta^{d-1}$$

is a bijection of the lattice $\mathbb{Z}^d$ on the ring $\mathbb{Z}[\beta]$.

In the following, we shall consider β an irrational Pisot number. Important property that will be used is the inclusion

$$\mathbb{Z}_\beta \subset \mathbb{Z}[\beta]\,. \tag{4.7}$$

Let us recall that $\mathbb{Z}_\beta$ is a proper subset of $\mathbb{Z}[\beta]$, since $\mathbb{Z}[\beta]$ is dense in $\mathbb{R}$ as a projection of the lattice $\mathbb{Z}^d$, whereas $\mathbb{Z}_\beta$ has no accumulation points. Since $\mathbb{Z}[\beta]$ is a ring,

$$\mathbb{Z}_\beta + \mathbb{Z}_\beta \subset \mathbb{Z}[\beta] \qquad \text{and} \qquad \mathbb{Z}_\beta \cdot \mathbb{Z}_\beta \subset \mathbb{Z}[\beta]\,.$$

Consider an $x \in \mathbb{Z}_\beta$ with the β-expansion $x = \sum_{k=0}^{n} a_k\beta^k$. Then

$$|x^{(i)}| = \left|\sum_{k=0}^{n} a_k (\beta^{(i)})^k\right| < \sum_{k=0}^{\infty} \lfloor\beta\rfloor \left|\beta^{(i)}\right|^k = \frac{\lfloor\beta\rfloor}{1 - |\beta^{(i)}|}\,,$$

for every $i = 2, 3, \ldots, d$. Therefore we can define

$$H_i := \sup\{|x^{(i)}| \mid x \in \mathbb{Z}_\beta\}\,. \tag{4.8}$$

To the inclusion (4.7) thus can be given precision

$$\mathbb{Z}_\beta \subset \{x \in \mathbb{Z}[\beta] \mid |x^{(i)}| < H_i,\ i = 2, 3, \ldots, d\}\,.$$

Another important property needed for determination of bounds on $L_\oplus$, $L_\otimes$ is finiteness of the set

$$C(l_1, l_2, \ldots, l_d) := \{x \in \mathbb{Z}[\beta] \mid |x| < l_1,\ |x^{(i)}| < l_i,\ i = 2, 3, \ldots, d\}\,,$$

for every choice of positive $l_1, l_2, \ldots, l_d$. A point $a_0 + a_1\beta + \cdots + a_{d-1}\beta^{d-1}$ belongs to $C(l_1, l_2, \ldots, l_d)$ only if the point $(a_0, a_1, \ldots, a_{d-1})$ of the lattice $\mathbb{Z}^d$ has all coordinates in the basis $\vec{y}_1$, $\ldots$, $\vec{y}_d$ in a bounded interval $(-l_i, l_i)$, i.e. $(a_0, a_1, \ldots, a_{d-1})$ belongs to a centrally symmetric parallelepiped. Every parallelepiped contains only finitely many lattice points.

Let us mention that notation $C(l_1, l_2, \ldots, l_d)$ is kept in accordance with [1], where Akiyama finds some conditions for $\mathrm{Fin}(\beta)$ to be a ring according to the properties of $C(l_1, l_2, \ldots, l_d)$. Our aim here is to use the set for determining the bounds on the length of the β-fractional part of the results of additions and multiplications in $\mathbb{Z}_\beta$.

Theorem 4.5.1. *Let β be a Pisot number of degree d, and let H_2, H_3, $\dots$, H_d be defined by* (4.8)*. Then*

$$\begin{aligned} L_\oplus &\leq \max\{\mathrm{fp}(r) \mid r \in \mathrm{Fin}(\beta) \cap C(1, 3H_2, 3H_3, \dots, 3H_d)\}, \\ L_\otimes &\leq \max\{\mathrm{fp}(r) \mid r \in \mathrm{Fin}(\beta) \cap C(1, H_2^2 + H_2, \dots, H_d^2 + H_d)\}. \end{aligned}$$

Proof. Consider $x, y \in \mathbb{Z}_\beta$ such that $x + y > 0$, $x + y \in \mathrm{Fin}(\beta)$. If we set

$$z := \max\{w \in \mathbb{Z}_\beta \mid w \leq x + y\},$$

then $r := x + y - z$ is the β-fractional of $x + y$ and thus

1. $r \in \mathrm{Fin}(\beta)$
2. $\mathrm{fp}(r) = \mathrm{fp}(x + y)$
3. $0 \leq r < 1$.

The numbers x, y, z belong to the ring $\mathbb{Z}[\beta]$ and hence also $r \in \mathbb{Z}[\beta]$. From the triangle inequality

$$|r^{(i)}| = |x^{(i)} + y^{(i)} - z^{(i)}| \leq 3H_i$$

for all $i = 2, 3, \dots, d$. Therefore r belongs to the finite set $C(1, 3H_2, 3H_3, \dots, 3H_d)$, which together with the definition of $L_\oplus$ gives the statement of the theorem for addition.

The upper bound on $L_\otimes$ is obtained analogically, with one change – because r is defined as $r := x \cdot y - z$ the triangular inequality gives us

$$|r^{(i)}| = |x^{(i)} \cdot y^{(i)} - z^{(i)}| \leq H_i^2 + H_i.$$

□

4.6 Application to β solution of $x^3 = 25x^2 + 15x + 2$

We apply the above Theorem 4.5.1 on $\beta > 1$ solution of the equation

$$x^3 = 25x^2 + 15x + 2.$$

Recall that such β satisfies the conditions of Proposition 4.4.1 for both conjugates β', β'' and thus Theorem 4.1.1 cannot be used for determining the bounds on $L_\oplus$, $L_\otimes$.

The Rényi development of 1 is $d_\beta(1) = (25)(15)(2)$. Since the minimal polynomial of β satisfies the assumptions of Theorem 3.1.3, the set $\mathrm{Fin}(\beta)$ is a ring.

In case that some of the algebraic conjugates of β is a real number, the bounds from Theorem 4.5.1 can be refined. In our case β is totally real. Let $x \in \mathbb{Z}_\beta$, $x = \sum_{i=0}^n a_i \beta^i$. Since $\beta' < 0$, we have

$$x' = \sum_{i=0}^n a_i (\beta')^i \leq \sum_{i=0, i \text{ even}}^n a_i (\beta')^i < \sum_{i=0}^\infty (25)(\beta')^{2i} = \frac{25}{1 - \beta'^2} =: H_1 \,.$$

The lower bound on x' is

$$x' = \sum_{i=0}^{n} a_i(\beta')^i \geq \sum_{i=0,i \text{ odd}}^{n} a_i(\beta')^i > \beta' H_1 \,.$$

Similarly for x'' we obtain

$$\beta'' H_2 < x'' < \frac{25}{1-\beta''^2} =: H_2 \,.$$

Consider $x, y \in \mathbb{Z}_\beta$ such that $x+y>0$. Again, the β-fractional part of $x+y$ has the form $r = x+y-z$ for some $z \in \mathbb{Z}_\beta$. Thus

$$\begin{gathered}(2\beta'-1)H_1 = \beta' H_1 + \beta' H_1 - H_1 < r' = x'+y'-z' < H_1 + H_1 - \beta' H_1 = (2-\beta')H_1 \\ (2\beta''-1)H_2 < r'' = x''+y''-z'' < (2-\beta'')H_2\end{gathered}$$

We have used a computer to calculate explicitly the set of remainders $r = A+B\beta+C\beta^2$, $A, B, C \in \mathbb{Z}$, satisfying

$$\begin{array}{rcccl} 0 < & & A+B\beta+C\beta^2 & & <1 \\ (2\beta'-1)H_1 & < & A+B\beta'+C\beta'^2 & < & (2-\beta')H_1 \\ (2\beta''-1)H_2 & < & A+B\beta''+C\beta''^2 & < & (2-\beta'')H_2 \end{array}$$

where for β', β'' we use numerical values (see Example in Section 4.4). The set has 93 elements, which we shall not list here (see Appendix C). For every element of the set we have found the corresponding β-expansion. The maximal length of the β-fractional part is 5. Thus $L_\oplus \leq 5$.

On the other hand, using the algorithm described in Section 3.1 we have found a concrete example of addition of numbers

$$\begin{aligned} (x)_\beta &= (25)(0)(25) \\ (y)_\beta &= (25)(0)(25) \end{aligned}$$

so that $x+y$ has the β-expansion

$$(x+y)_\beta = (1)(24)(12)(11) \bullet (23)(0)(14)(13)(2).$$

Thus we have found the value

$$L_\oplus = 5 \,.$$

In order to obtain bounds on $L_\otimes$, we have computed the list of all $r = A+B\beta+C\beta^2$, $A, B, C \in \mathbb{Z}$, satisfying the inequalities

$$\begin{array}{rcccl} 0 < & & A+B\beta+C\beta^2 & & <1 \\ \beta' H_1^2 - H_1 & < & A+B\beta'+C\beta'^2 & < & H_1^2 - \beta' H_1 \\ \beta'' H_2^2 - H_2 & < & A+B\beta''+C\beta''^2 & < & H_2^2 - \beta'' H_2 \end{array}$$

In this case we have obtained 8451 candidates on the β-fractional part of multiplication. The longest one among them has 7 digits. On the other hand for numbers $(x)_\beta = (25)$ and $(y)_\beta = (25)$ we have

$$(x \cdot y)_\beta = (24)(10) \bullet (21)(24)(16)(7)(16)(13)(2).$$

Therefore

$$L_\otimes = 7\,.$$

Let us mention that the above method can be applied also to the case of Tribonacci number, but the bounds obtained in this way are not better than those from Theorem 4.1.1. We get $L_\oplus \le 6$, $L_\otimes \le 6$.

Chapter 5

Conclusion

The main aim of this work was to find methods for making estimates on the maximum of fractional digits that can arise under arithmetic operations with integers in systems with irrational base, i.e. the estimates on the values of coefficients $L_\oplus$ and $L_\otimes$.

Very important role in the searching of values of those coefficients plays the question whether or not the set $\mathrm{Fin}(\beta)$ – the set of all integers in a given numeration system – is a ring. So that was our first task – to find some conditions for $\mathrm{Fin}(\beta)$ being a ring.

The first proposition proved was an auxiliary one, which facilitated our work, asserting that it is enough study only the question of addition of positive numbers in order to decide about closure of $\mathrm{Fin}(\beta)$.

Under this simplification we found and proved a necessary condition for this closure of the set $\mathrm{Fin}(\beta)$ – called Property T. Further, we found one more condition for the number β satisfying Property T, which is sufficient to $\mathrm{Fin}(\beta)$ being closed under addition of positive elements.

While proving the theorem about sufficient condition, we gave an algorithm for performing arithmetic operations in systems fulfilling the assumptions of this theorem. This algorithm was afterwards used to write an computer program `BetaArithmetic` – very useful for finding the lower bounds on $L_\oplus$ and $L_\otimes$. One of results of this search for lower bounds on these coefficients was an example of multiplication in the Tribonacci numeration system giving four fractional digits. Hence we refute the conjecture of Arnoux, see [9], that $L_\otimes = 3$ for Tribonacci multiplication.

In the second part of this work – the estimation of $L_\oplus$ and $L_\otimes$ itself – we showed two methods suitable for coping with this task. The first one is applicable to algebraic numbers $\beta > 1$ with at least one conjugate β' satisfying

$$\sup\{|z'| \mid z \in \mathbb{Z}_\beta\} < +\infty$$
$$\inf\{|z'| \mid z \in \mathbb{Z}_\beta \setminus \beta\mathbb{Z}_\beta\} > 0$$

where $\mathbb{Z}_\beta$ is the set of all integers in a given numeration system, whereas the second methods needs β to be a Pisot number.

We used these two methods to find upper bounds on $L_\oplus$ and $L_\otimes$ for minimal Pisot numeration system, Tribonacci numeration system and 22-15-2 Pisot numeration system. In the case of Tribonacci multiplication we found better bound on $L_\otimes$ than 9 – the value formerly found by Messaoudi [9].

Nevertheless there are still open problems to be solved, some new ones also appeared while we were writing this work. As some of the most obvious we can point out following three:

1. It is clear from the second method of estimate that for β a Pisot number $L_\oplus$, $L_\otimes$ are finite numbers. Does there exist a β non Pisot such that $L_\oplus = +\infty$ or $L_\otimes = +\infty$?
2. Whenever β satisfies Property T, it is possible to apply repeatedly the transcription on the β-representation of $x + y$, $x, y \in \mathrm{Fin}(\beta)$, $x, y > 0$. If the transcription can be applied infinitely many times, what is the order of choice of forbidden strings so that the sequence of β-representations converges rapidly to the β-expansion of $x + y$?
3. It is known [12] that for β a Pisot number every $x \in \mathbb{Q}[\beta]$ has a finite or eventually periodic β-expansion. It would be interesting to have algorithms working with periodic expansions.

Appendix A

$\mathbf{Min}_n$ and $\mathbf{Max}_n$ for minimal Pisot number

In the following tables one can find complete list of consecutive search for the constants $\min_n$ and $\max_n$ in the minimal Pisot number numeration system.

They were used to obtain the upper bounds of the coefficients $L_\oplus$ and $L_\otimes$ (see Section 4.3). The coefficients for $n = 37$ was good enough to find the exact value of $L_\oplus$. For finding the estimate of the coefficient $L_\otimes \leq 13$ is was enough to consider $n = 35$, at the same time the lower bound is only $L_\otimes \geq 8$. We tried to find better upper estimate by increasing n. For $n = 36, \ldots, 60$ the estimate remained the same, unfortunately it takes huge amount of computations as well as disk space to further increase n.

constant	**value**	**$\boldsymbol{\beta}$-expansion**
$\min_{20}$	0.557061	10000100001000000001
$\min_{21}$	0.557061	10000100001000000001
$\min_{22}$	0.557061	10000100001000000001
$\min_{23}$	0.550579	10000000100001000000001
$\min_{24}$	0.550579	10000000100001000000001
$\min_{25}$	0.550579	10000000100001000000001
$\min_{26}$	0.542378	10000010000100001000000001
$\min_{27}$	0.542378	10000010000100001000000001
$\min_{28}$	0.528962	1000010000000100001000000001
$\min_{29}$	0.528962	1000010000000100001000000001
$\min_{30}$	0.528962	1000010000000100001000000001
$\min_{31}$	0.528962	1000010000000100001000000001
$\min_{32}$	0.528962	1000010000000100001000000001
$\min_{33}$	0.520206	100001000010000000100001000000001
$\min_{34}$	0.520206	100001000010000000100001000000001
$\min_{35}$	0.520206	100001000010000000100001000000001
$\min_{36}$	0.520206	100001000010000000100001000000001
$\min_{37}$	0.520206	100001000010000000100001000000001

Table A.1: $\min_n$ for minimal Pisot number numeration system

constant	**value**	**$\boldsymbol{\beta}$-expansion**
max_{20}	1.75490	1000000010000100001
max_{21}	1.77188	100001000010000100001
max_{22}	1.77188	100001000010000100001
max_{23}	1.77188	100001000010000100001
max_{24}	1.79286	100001000000010000100001
max_{25}	1.79286	100001000000010000100001
max_{26}	1.79286	100001000000010000100001
max_{27}	1.79286	100001000000010000100001
max_{28}	1.79286	100001000000010000100001
max_{29}	1.80847	10000100001000000010000100001
max_{30}	1.80847	10000100001000000010000100001
max_{31}	1.80847	10000100001000000010000100001
max_{32}	1.80847	10000100001000000010000100001
max_{33}	1.80847	10000100001000000010000100001
max_{34}	1.81356	1000010000100001000000010000100001
max_{35}	1.81356	1000010000100001000000010000100001
max_{36}	1.81356	1000010000100001000000010000100001
max_{37}	1.81481	1000000010000100001000000010000100001

Table A.2: $\max_n$ for minimal Pisot number numeration system

Appendix B

User's guide to program BetaArithmetic

In this appendix, we present a brief user's guide to the program BetaArithmetic. This program performs arithmetic operations in Pisot numeration systems with base β, where β is a greatest root of the equation $x^m = a_{m-1}x^{m-1} + \cdots + a_1x + a_0$, where $a_{m-1} \geq \cdots \geq a_1 \geq a_0 \geq 1$ and $m \leq 8$.

There are three different implementation of this program, available on the companion CD-ROM. The first one is a workstation version for Win32 systems accompanied with GUI[1]. The second version is a CGI[2] version prepared to run on a web server to be accessible over the internet. The window of the workstation version for Win32 system is very similar to the HTML page generated by the CGI version, hence the description of the functionality is the same for both of these versions. The third version is a command line version prepared to perform bath tasks and can be compiled on any system having ANSI C++ compiler.

B.1 Command line version

The syntax for running the command line has the following form

```
BetaArithmetic {+,*} [options]
options: -i1 'filename'
         -i2 'filename'
         -o 'filename'
         -d 'n'
         -v
```

The first parameter on the command line has to be either `+` or `*` to determine the arithmetic operation. Additional (optional) parameters are

`-i1 'filename'` `'filename'` is a name of the first input file
`-i2 'filename'` `'filename'` is a name of the second input file

[1]Graphical user interface
[2]Common gateway interface

these two parameters complement each other, therefore there are only two possibilities: one can use either none of them or both of them.

`-o 'filename'`	`'filename'` is a name of the output file
`-d 'n'`	`'n'` is a number of fractional digits a result has to have to be printed on the output
`-v`	output is printed in the simplified form

the simplified form of the output means, that only the resultant number is printed, but nothing else.

B.2 GUI version

The version of the program accompanied with the GUI (as well as the CGI version) do not perform any batch tasks, it is prepared to perform the computations one by one as the user inputs data.

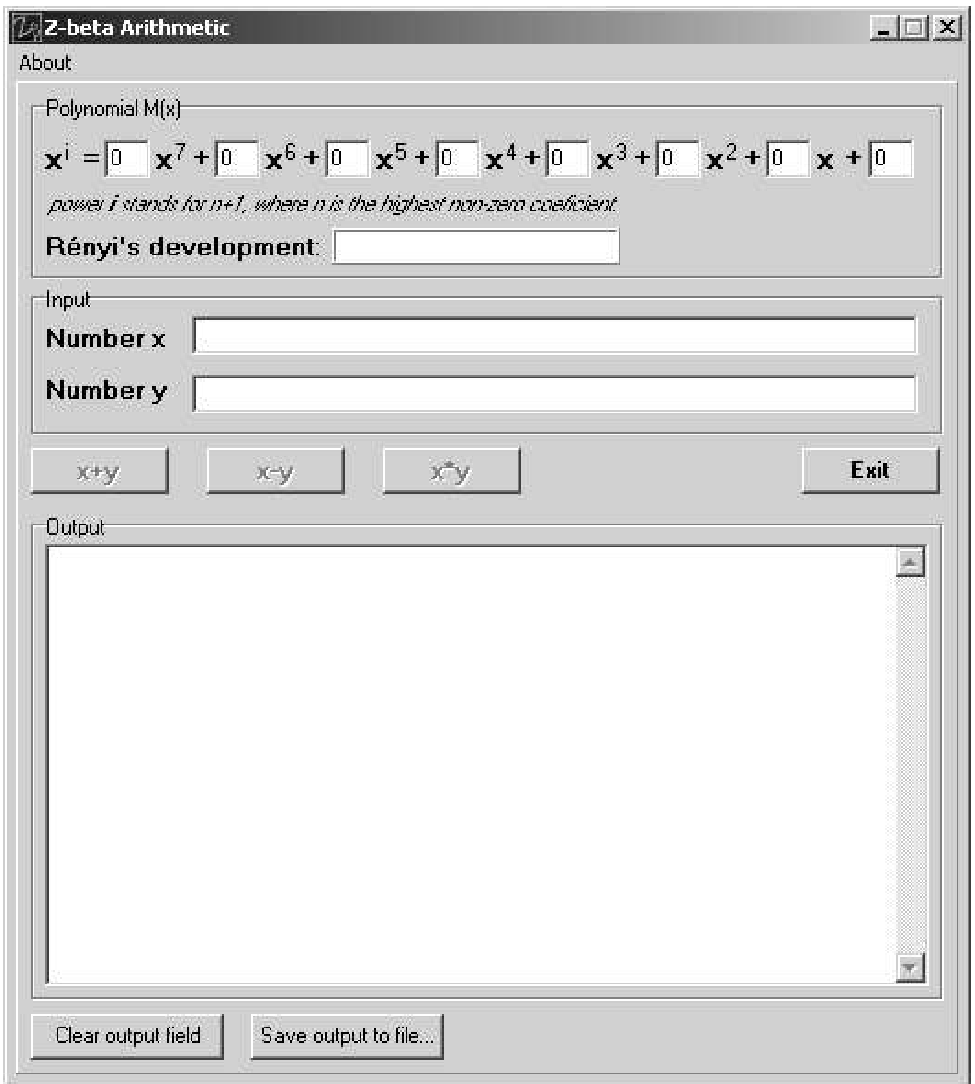

Figure B.1: The window of the program BetaArithmetic

All the computation are controlled within the main window (Figure B.2). The parameters a user has to input to get an result are divided into two parts.

In the first part (denoted `Polynomial M(x)`) there are boxes for assignment of the coefficients of the minimal polynomial of the base β. If the coefficients are non-increasing, the Renyi development of unit is in respective box, otherwise there is only red exclamation mark.

The second part (denoted `Input numbers`) there are boxes of numbers the users wants to add (subtract or multiply). Because there are numeration systems where coefficients in β-expansion can be greater than nine, it is required to separate the coefficients with a gap, the fractional mark and possible minus mark has to be separated by a gap too.

Example: the input string

```
- 25 0 25 .  0 3
```

stands for the number $-(25\beta^2 + 25 + 3\beta^{-2})$.

Appendix C

Content of the CD-ROM disc

This appendix contains the description of the directory structure on the companion CD-ROM.

directory	description
\Programs	
\Bin	*compiled binaries of all programs*
\Src	*source files of all programs*
\BetaArtihmetic.ANSI	*command-line version performing batch tasks*
\BetaArtihmetic.CGI	*CGI version for a web-server*
\BetaArtihmetic.Win32	*workstation version with GUI for Win32 systems*
\HKFinder	*tool for using the H-K method*
\MaxFinder	*tool for the analysis of output files in batch tasks*
\Data	
\Input	*input files*
\Minimal	*β-expansions in minimal Pisot number numeration system*
\Tribonacci	*β-expansions in Tribonacci number numeration system*
\Output	*results of computations*
\Minimal	*H-K method for minimal Pisot number numeration system*
\25-15-2	*2nd method for 25-15-2 Pisot number numeration system*

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