

On alpha-adic expansions in Pisot bases¹

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Abstract

We study α -adic expansions of numbers, that is to say, left infinite representations of numbers in the positional numeration system with the base α , where α is an algebraic conjugate of a Pisot number β . Based on a result of Bertrand and Schmidt, we prove that a number belongs to $\mathbb{Q}(\alpha)$ if and only if it has an eventually periodic α -expansion. Then we consider α -adic expansions of elements of the ring $\mathbb{Z}[\alpha^{-1}]$ when β satisfies the so-called Finiteness property (F). We give two algorithms for computing these expansions — one for positive and one for negative numbers. In the particular case that β is a quadratic Pisot unit satisfying (F), we inspect the unicity and/or multiplicity of α -adic expansions of elements of $\mathbb{Z}[\alpha^{-1}]$. We also provide algorithms to generate α -adic expansions of rational numbers in that case.

1 Introduction

Most usually, real numbers are represented in a positional numeration systems, that is, numbers are considered in the form of finite or infinite words over a given ordered set — an alphabet of digits, and their value is taken following the powers of a real base $\beta > 1$. Several different types of these systems have been studied in the past, e.g. usual representations in an integer base (and its generalizations such as p -adic numeration or systems using signed digits), representations in an irrational base, based on the so-called

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β -expansions (introduced by Rényi [23]), or representations with respect to a sequence of integers, like the Fibonacci numeration system. Another approach is also canonical number systems as studied in [18]) for instance. A survey of most of these concepts was given in [19, Chapter 7].

In this paper we study another way of representing numbers, strongly connected with the above mentioned representations based on β -expansions. It is called the α -adic representation and, roughly speaking, is a representation of a complex (or real) number in a form of (possibly) left infinite power series in α , where α is a complex (or real) number of modulus less than 1.

We have two sources of inspiration — the β -numeration systems on one hand and the p -adic numbers (representations of numbers in the form of left infinite power series in a prime p) on the other hand. However, contrary to the usual p -adic numbers the base of the α -adic system is taken to be in modulus smaller than one. This fact implies an important advantage over the usual p -adic expansions, since we do not have to introduce any special valuation for the series to converge.

In β -expansions, numbers are right infinite power series. The deployment of left infinite power series has been used by several authors for different purposes. Vershik [28] (probably the first use of the term fibadic expansion) and Sidorov and Vershik [27] use two-sided expansions to show a connection between symbolic dynamics of toral automorphisms and arithmetic expansions associated with their eigenvalues and for study of the Erdős measure (more precisely two-sided generalization of Erdős measure). Two-sided beta-shifts have been studied in full generality by Schmidt in [26]. Ito and Rao [17], and Berthé and Siegel [7] use representations of two-sided β -shift in their study of purely periodic expansions with Pisot unit and non-unit base. The realization by a finite automaton of the odometer on the two-sided β -shift has been studied by Frougny [13].

Left-sided extensions of numeration systems defined by a sequence of integers, like the Fibonacci numeration system, have been introduced by Grabner, Liardet and Tichy [15], and studied from the point of view of the odometer function. The use (at least implicit) of representations infinite to the left is contained in every study of the Rauzy fractal [22], especially in a study of its border, see e.g. Akiyama [1], Akiyama and Sadahiro [4], or Messaoudi [20]. Finally, there is a recent paper by Sadahiro [24] on multiply covered points in the conjugated plane in the case of cubic Pisot units having complex conjugates. Sadahiro's approach to the left infinite expansions is among all mentioned works the closest one to our own.

This contribution is organized as follows. First, we recall known facts about β -numeration and we define α -adic expansions in the case where α is an algebraic

conjugate of a Pisot number β . Recall that, by the results of Bertrand [8] and Schmidt [25], a positive real number belongs to the extension field $\mathbb{Q}(\beta)$ if and only if its β -expansion (which is right infinite) is eventually periodic. Thus it is natural to try to get a similar result for the α -adic expansions where α is an algebraic conjugate of β . We prove that a number belongs to the field $\mathbb{Q}(\alpha)$ if and only if its α -adic expansion is eventually periodic to the left with a finite α -fractional part. Note that the fields $\mathbb{Q}(\alpha)$ and $\mathbb{Q}(\beta)$ are identical (up to an isomorphism), but our result includes also negative numbers; that means one can represent by α -adic expansions with positive digits also negative numbers without utilization of the sign.

Further on, we consider α -adic expansions of elements of the ring $\mathbb{Z}[\alpha^{-1}]$ in the case when β satisfies the Finiteness property (F). We give two algorithms for computing these expansions — one for positive and one for negative numbers.

Finally, in the case of quadratic Pisot units satisfying (F), we study the unicity of the expansions of elements of the ring $\mathbb{Z}[\alpha^{-1}]$. We give an algorithm for computing an α -adic representation of a rational number and we discuss the normalization of such a representation by means of a finite transducer.

2 Preliminaries

2.1 Words

An *alphabet* A is a finite ordered set. We denote by A^* the set of all *finite words* over A , i.e. the set of finite concatenation of letters from A . The empty word (identity of the *free monoid* A^*) is denoted by ε . The set of (right) *infinite words* on A is denoted by $A^{\mathbb{N}}$. A word $u \in A^{\mathbb{N}}$ is said to be *eventually periodic* if it is of the form $u = vz^\omega$, where $v, z \in A^*$ are finite words and $z^\omega = zzz \dots$ denotes the infinite concatenation of z to itself. We consider also left infinite words, and the set of these words on A is denoted ${}^{\mathbb{N}}A$. A word $u \in {}^{\mathbb{N}}A$ is eventually periodic if $u = {}^\omega zv$, where $v, z \in A^*$ and ${}^\omega z = \dots zzz$. A *factor* of a (finite or infinite) word u is a finite word v such that $u = v_1vv_2$ for some words v_1, v_2 .

2.2 Automata and transducers

An *automaton* over an alphabet A , denoted $\mathcal{A} = \langle A, Q, E, I, F \rangle$, is a directed graph with labels in A . The set Q is the set of its vertices, called *states*, $I \subset Q$ is the set of *initial states*, $F \subset Q$ is the set of *final states* and $E \subset Q \times A \times Q$ is

the set of labeled edges, called *transitions*. The automaton is said to be *finite* if the set of states Q is finite. If $(p, a, q) \in E$ one usually writes $p \xrightarrow{a} q$.

A *computation* c in \mathcal{A} is a finite sequence of transitions such that

$$c = q_0 \xrightarrow{a_1} q_1 \xrightarrow{a_2} q_2 \xrightarrow{a_3} \cdots \xrightarrow{a_n} q_n .$$

The *label* of the computation c is the finite word $a_1 a_2 \cdots a_n$ of A^* . The computation c is *successful* if $q_0 \in I$ and $q_n \in F$. The *behavior* of \mathcal{A} , denoted by $|\mathcal{A}|$, is the subset of A^* of labels of all successful computations of \mathcal{A} . An automaton \mathcal{A} is called *deterministic* if for any pair $(p, a) \in Q \times A$ there exist at most one state $q \in Q$ such that $p \xrightarrow{a} q$ is a transition of \mathcal{A} , and if there is only one initial state.

An automaton $\mathcal{T} = \langle A^* \times B^*, Q, E, I, F \rangle$ over a non-free monoid $A^* \times B^*$ is called a *transducer* from A^* to B^* . Its transitions are labeled by pairs of words $(u, v) \in A^* \times B^*$, the word u is called *input* and the word v is called *output*. If $(p, (u, v), q) \in E$ one usually writes $p \xrightarrow{u|v} q$. The transducer \mathcal{T} is finite if Q and E are finite.

A computation c in \mathcal{T} is a finite sequence

$$c = q_0 \xrightarrow{u_1|v_1} q_1 \xrightarrow{u_2|v_2} q_2 \xrightarrow{u_3|v_3} \cdots \xrightarrow{u_n|v_n} q_n .$$

The label of the computation c is $(u, v) := (u_1 u_2 \cdots u_n, v_1 v_2 \cdots v_n)$. The behavior of a transducer \mathcal{T} is a relation $R \subset A^* \times B^*$. If for any word $u \in A^*$ there exists at most one word $v \in B^*$ such that $(u, v) \in R$, the transducer is said to *compute* (*realize*) a function. A transducer is called *real-time* if input words of all its transitions are letters in A (*i.e.* the transitions are labeled in $A \times B^*$). The *underlying input* (respectively *output*) *automaton* of a transducer \mathcal{T} is obtained by omitting the output (respectively input) labels of each transition of \mathcal{T} . A transducer is said to be *sequential* if it is real-time and its underlying input automaton is deterministic. A function is called *sequential* if it can be realized by a sequential transducer.

All these definitions implicitly suppose that words are processed as usual from left to right. The same definitions can be done where words are processed from right to left; then we speak of *right* automata or transducers.

2.3 Beta-expansions

Let $\beta > 1$ be a real number. A *representation in base β* (or simply a *β -representation*) of a real number $x \in \mathbb{R}_+$ is an infinite sequence $(x_i)_{i \leq k}$, such

that $x_i \in \mathbb{Z}$ and

$$x = x_k\beta^k + x_{k-1}\beta^{k-1} + \cdots + x_1\beta + x_0 + x_{-1}\beta^{-1} + x_{-2}\beta^{-2} + \cdots$$

for a certain $k \in \mathbb{Z}$. If a β -representation of x ends in infinitely many zeros, it is said to be *finite* and the ending zeros are omitted.

A particular β -representation — called the β -*expansion* [23] — is computed by the so-called greedy algorithm. Denote by $\lfloor y \rfloor$, respectively by $\{y\}$, the integer part, respectively the fractional part, of a number y . Let x be a non-negative real number. Find $k \in \mathbb{Z}$ such that $\beta^k \leq x < \beta^{k+1}$. Set $x_k := \lfloor x/\beta^k \rfloor$ and $r_k := \{x/\beta^k\}$ and let for $i < k$, $x_i = \lfloor \beta r_{i+1} \rfloor$ and $r_i = \{\beta r_{i+1}\}$. The sequence $(x_i)_{i \leq k}$ is said to be the β -*expansion* of x . It is denoted

$$\langle x \rangle_\beta = x_k x_{k-1} \cdots x_0 \bullet x_{-1} x_{-2} \cdots ,$$

most significant digit first. When k is negative, we set $x_{-1} = \cdots = x_{-k+1} = 0$. If β is not an integer, the digits x_i obtained by the greedy algorithm are elements of the alphabet $A = \{0, 1, \dots, \lfloor \beta \rfloor\}$, called the *canonical alphabet*. The β -expansion is the greatest β -representation of a number in the lexicographic order.

The word $x_k x_{k-1} \cdots x_0$ at the left of the radix point of the β -expansion of x is called the β -*integer part* of x , and the infinite word $x_{-1} x_{-2} \cdots$ at the right of the radix point is called the β -*fractional part* of x . When the context is clear, the symbol “ β ” is omitted.

Let $x_k x_{k-1} \cdots x_0 \bullet x_{-1} x_{-2} \cdots$ be a β -representation. The β -*value* is the function $\pi_\beta : A^\mathbb{N} \rightarrow \mathbb{R}$ defined by $\pi_\beta(x_k x_{k-1} \cdots) := \sum_{k \geq i} x_i \beta^i$.

Let C be a finite alphabet of integers. The *normalization* on C is the function $\nu_C : C^\mathbb{N} \rightarrow A^\mathbb{N}$ that maps a word $w = (w_i)_{i \leq k}$ of $C^\mathbb{N}$ onto $\langle x \rangle_\beta$, where $x = \sum_{i \leq k} w_i \beta^i$, that is to say, it maps a β -representation of a number x onto its β -expansion. A reader interested in possible technical realizations of the normalization is referred to [14, Theorem 2], or [6, Section 3.4].

Recall that a *Pisot number* is an algebraic integer $\beta > 1$ whose algebraic conjugates are in modulus less than one.

Theorem 1 ([12]) *If β is a Pisot number then on any alphabet C of digits the normalization function $\nu_C : C^\mathbb{N} \rightarrow A^\mathbb{N}$ is computable by a finite transducer with edges labelled by elements of $C \times A$.*

A sequence of coefficients which corresponds to some β -expansion is usually called *admissible* in the β -numeration system. For the characterization of admissible sequences we use Parry’s condition [21]. Let $T_\beta : [0, 1] \rightarrow [0, 1)$ be the β -*transformation* on the unit interval defined by $T_\beta(x) := \{\beta x\}$. The sequence

$d_\beta(1) = t_1 t_2 t_3 \cdots$ such that $t_i = \lfloor \beta T_\beta^{i-1}(1) \rfloor$ is called the *Rényi expansion* of 1. If $d_\beta(1)$ is finite and ℓ is the greatest index such that $t_\ell \neq 0$ in $d_\beta(1)$ we set $d_\beta^*(1) = (t_1 t_2 \cdots t_{\ell-1} (t_\ell - 1))^\omega$, otherwise $d_\beta(1)$ has infinitely many non-zero digits t_i and we set $d_\beta^*(1) = d_\beta(1)$.

Theorem 2 (Parry [21]) *An infinite sequence $(x_i)_{i \leq k}$ is the β -expansion of a real number $x \in [0, 1)$ if and only if for all $j \leq k$ the sequence $x_j x_{j-1} x_{j-2} \cdots$ is strictly lexicographically smaller than the sequence $d_\beta^*(1)$.*

Properties of β -expansions are strongly related to symbolic dynamics. The closure of the set of admissible β -expansions is called the β -shift. It is a symbolic dynamical system, that is, a closed shift-invariant subset of $A_\beta^{\mathbb{N}}$, see the surveys [9] and [19, Chapter 7] for more details.

Let $\mathbb{Q}(\beta)$ be the minimal subfield of complex numbers \mathbb{C} containing all rationals \mathbb{Q} as well as the algebraic number β . Let α be an algebraic conjugate of β , then the fields $\mathbb{Q}(\beta)$ and $\mathbb{Q}(\alpha)$ are isomorphic and their isomorphism is induced by the assignment $\beta \mapsto \alpha$. Formally, the isomorphism $' : \mathbb{Q}(\beta) \rightarrow \mathbb{Q}(\alpha)$ is defined by setting $(g(\beta))' = g(\alpha)$, where $g(X)$ is a polynomial in X with rational coefficients.

There is a nice characterization of β -expansions of elements of $\mathbb{Q}(\beta)$ due independently to Bertrand [8] and Schmidt [25].

Theorem 3 *Let β be a Pisot number. A positive real number x has an eventually periodic β -expansion if and only if $x \in \mathbb{Q}(\beta)$.*

2.4 The Finiteness property (F)

The set of all real numbers x for which the β -expansion of $|x|$ is finite is denoted by $\text{Fin}(\beta)$. A number β is said to *satisfy Property (F)* if

$$\text{Fin}(\beta) = \mathbb{Z}[\beta^{-1}].$$

It has been proved [14] that Property (F) implies that β is a Pisot number and that $d_\beta(1)$ is finite. Conversely, to find a simple algebraic characterization of Pisot numbers satisfying (F) is an open problem up to now. Let

$$M_\beta(X) = X^d - a_{d-1}X^{d-1} - \cdots - a_1X - a_0 \tag{1}$$

be the minimal polynomial of an algebraic integer β . Several authors have found some sufficient conditions on $M_\beta(X)$ for β to have Property (F).

Theorem 4 ([14]) *If the coefficients in (1) fulfill $a_{d-1} \geq a_{d-2} \geq \cdots \geq a_1 \geq a_0 > 0$, then β has Property (F).*

Theorem 5 (Hollander [16]) *If the coefficients in (1) fulfill $a_{d-1} > a_{d-2} + \dots + a_1 + a_0$ with $a_i \geq 0$, then β has Property (F).*

Theorem 6 (Akiyama [2]) *Let β be a cubic Pisot unit. Then β has Property (F) if and only if the coefficients in (1) fulfill $a_0 = 1$, $a_2 \geq 0$ and $-1 \leq a_1 \leq a_2 + 1$.*

Note that in [3] is studied a weaker property, called the *weak finiteness property* (W). In particular it is proved that every cubic Pisot unit satisfies Property (W), and it is conjectured that all Pisot numbers satisfy it.

2.5 Alpha-adic expansions

From now on let β be a Pisot number with finite Rényi expansion of 1, say $d_\beta(1) = t_1 \cdots t_\ell$. Let α be one of its algebraic conjugates.

Definition 7 *An α -adic representation of a number $x \in \mathbb{C}$ is a left infinite sequence $(x_i)_{i \geq -k}$ such that $x_i \in \mathbb{Z}$ and*

$$x = \cdots + x_2 \alpha^2 + x_1 \alpha + x_0 + x_{-1} \alpha^{-1} + \cdots + x_{-k} \alpha^{-k}$$

for a certain $k \in \mathbb{Z}$. It is denoted ${}_\alpha(x) = \cdots x_1 x_0 \bullet x_{-1} \cdots x_{-k}$.

Definition 8 *A (finite, right infinite or left infinite) sequence is said to be weakly admissible if all its finite factors are lexicographically less than or equal to $d_\beta^*(1)$, which is equivalent to the fact that each factor of length ℓ is less than $t_1 \cdots t_\ell$ in the lexicographic order.*

If an α -adic representation $(x_i)_{i \geq -k}$ of a number x is weakly admissible it is said to be an α -adic expansion of x , denoted ${}_\alpha \langle x \rangle = \cdots x_1 x_0 \bullet x_{-1} \cdots x_{-k}$.

Example 9 *Let $\beta > 1$ be the golden mean, that is, the Pisot number with minimal polynomial $X^2 - X - 1$. Then $d_\beta(1) = 11$ and $d_\beta^*(1) = (10)^\omega$. Hence 11 is a forbidden factor in any β -expansion or α -adic expansion. Moreover the sequence $(10)^\omega$ is a forbidden tail in any β -expansion. On the other hand, ${}^\omega(10)010 \bullet 1$ is weakly admissible, and is an α -adic expansion of -2 .*

Remark 10 *Although the β -expansion of a number is unique, the α -adic expansion is not. For instance in the α -adic system associated with the golden mean, the number -1 has two α -adic expansions*

$$\begin{aligned} {}_\alpha \langle -1 \rangle &= {}^\omega(10) \bullet \\ {}_\alpha \langle -1 \rangle &= {}^\omega(10)0 \bullet 1 \end{aligned}$$

If ${}_{\alpha}\langle x \rangle = \cdots x_1 x_0 \bullet x_{-1} \cdots x_{-k}$ is an α -adic expansion of a number $x \in \mathbb{C}$, the left infinite word $\cdots x_1 x_0$ at the left of the radix point is called the α -integer part, and the word $x_{-1} \cdots x_{-k}$ at the right of the radix point is called the α -fractional part. When the context is clear, the symbol “ α ” is omitted.

Analogous to the case of β -representations we define for α -adic expansions the α -value function π_{α} and the normalization function ν_G .

3 Eventually periodic α -adic expansions

In order to prove the main theorem about eventually periodic expansions, we need two technical lemmas.

Lemma 11 *Let $y \in \mathbb{Q}(\beta) \cap (0, 1)$ be a real number with the purely periodic β -expansion $\langle y \rangle_{\beta} = 0 \bullet (y_{-1} \cdots y_{-p})^{\omega}$. Then its conjugate y' satisfies*

$${}_{\alpha}\langle -y' \rangle = {}^{\omega}(y_{-1} \cdots y_{-p}) \bullet 0.$$

PROOF. We have $y = \frac{y_{-1}}{\beta} + \cdots + \frac{y_{-p}}{\beta^p} + \frac{y_{-1}}{\beta^{p+1}} + \cdots$, which can be also written $y = \frac{y_{-1}}{\beta} + \cdots + \frac{y_{-p}}{\beta^p} + \frac{y}{\beta^p}$. Conjugating the equation we obtain $y' = \frac{y_{-1}}{\alpha} + \cdots + \frac{y_{-p}}{\alpha^p} + \frac{y'}{\alpha^p}$. Hence $-y' = y_{-1}\alpha^{p-1} + \cdots + y_{-p} - y'\alpha^p$ that is ${}_{\alpha}\langle -y' \rangle = {}^{\omega}(y_{-1} \cdots y_{-p}) \bullet 0$. \square

Lemma 12 *Let $x \in \mathbb{Q}(\beta) \cap (0, 1)$ be a real number with finite β -expansion $\langle x \rangle_{\beta} = 0 \bullet x_{-1} \cdots x_{-p}$, then ${}_{\alpha}\langle x' \rangle$ is of the form*

$${}^{\omega}(t_1 \cdots t_{\ell-1}(t_{\ell} - 1)) u_n \cdots u_0 \bullet u_{-1} \cdots u_{-m}.$$

PROOF. Let $\langle x \rangle_{\beta} = 0 \bullet x_{-1} \cdots x_{-p}$ with $x_{-p} \neq 0$. By conjugating it and by changing the sign of its coefficients we obtain an α -adic representation of $-x'$, ${}_{\alpha}\langle -x' \rangle = 0 \bullet \overline{x_{-1}} \cdots \overline{x_{-p}}$, where \overline{d} denotes the signed digit $-d$. If we subtract -1 from the last non-zero coefficient $\overline{x_{-p}}$ and replace it by an α -adic expansion of -1 of the form ${}_{\alpha}\langle -1 \rangle = {}^{\omega}(t_1 \cdots t_{\ell-1}(t_{\ell} - 1)) \bullet$, we obtain another representation, which is eventually periodic to the left with a pre-period of the form of a finite word over the alphabet $\{-\lfloor \beta \rfloor, \dots, \lfloor \beta \rfloor\}$. Finally, to obtain an α -adic expansion of $-x'$ we have to normalize the pre-period, that is to say, to find an admissible word numerically equivalent to the pre-period. Note that the pre-period can be seen as a difference between two finite expansions and so the normalization will not interfere with the period. \square

Lemma 11 and 12 allow us to derive from Theorem 3 a characterization of numbers with eventually periodic α -adic expansions. The main difference with Theorem 3 is that the version for α -adic expansions includes also negative numbers, that is, one can represent by α -adic expansions with positive digits also negative numbers without the necessity of utilization of the sign.

Theorem 13 *Let α be a conjugate of a Pisot number β . A number x in \mathbb{C} has an eventually periodic α -adic expansion if and only if $x \in \mathbb{Q}(\alpha)$.*

PROOF. \Leftarrow : Let x have an eventually periodic α -adic expansion, say ${}_{\alpha}\langle x \rangle = {}^{\omega}(x_{k+p} \cdots x_{k+1})x_k \cdots x_0 \bullet x_{-1} \cdots x_{-j}$. Let $u := \sum_{i=-j}^k x_i \alpha^i$ and $v := \sum_{i=k+1}^{k+p} x_i \alpha^i$. Then $u, v \in \mathbb{Z}[\alpha^{-1}]$ and

$$x = u + \frac{v}{1 - \alpha^p},$$

which proves the implication.

\Rightarrow : Let $x \in \mathbb{Q}(\beta) \cap [0, 1)$. According to Theorem 3 the β -expansion of x is eventually periodic, say $\langle x \rangle_{\beta} = \mathbf{0} \bullet x_{-1} \cdots x_{-n} (x_{-n-1} \cdots x_{-n-p})^{\omega}$. In the case where the period of $\langle x \rangle_{\beta}$ is empty, an eventually periodic α -adic expansion of $-x'$ is obtained by Lemma 12.

Let us assume that the period of $\langle x \rangle_{\beta}$ is non-empty and let us denote $y := \pi_{\beta}(\mathbf{0} \bullet (x_{-(n+1)} \cdots x_{-(n+p)})^{\omega})$, therefore $x = \frac{x_{-1}}{\beta} + \cdots + \frac{x_{-n}}{\beta^n} + \frac{y}{\beta^n}$

Conjugating the equation we obtain $x' = \frac{x_{-1}}{\alpha} + \cdots + \frac{x_{-n}}{\alpha^n} + \frac{y'}{\alpha^n}$, hence $-x' = -\frac{y'}{\alpha^n} - \frac{x_{-1}}{\alpha} - \cdots - \frac{x_{-n}}{\alpha^n}$. According to Lemma 11 we know how to obtain an α -adic expansion of $-y'$, hence an α -adic representation of $-x'$ can be obtained by digit wise addition

$$\begin{array}{ccccccc} {}^{\omega}(x_{-(n+1)} \cdots x_{-(n+p)})x_{-(n+1)} \cdots x_{-p} \bullet & x_{-(p+1)} & \cdots & x_{-(n+p)} & & & \\ & \bullet & (-x_{-1}) & \cdots & (-x_{-n}) & & \\ \hline {}^{\omega}(x_{-(n+1)} \cdots x_{-(n+p)})x_{-(n+1)} \cdots x_{-p} \bullet & (x_{-(p+1)} - x_{-1}) & \cdots & (x_{-(n+p)} - x_{-n}) & & & \end{array}$$

Therefore we have ${}_{\alpha}\langle -x' \rangle$ of the form ${}^{\omega}(c_1 \cdots c_p)u$, where u is a finite word, obtained by the normalization of the pre-period $x_{-(n+1)} \cdots x_{-p} \bullet (x_{-(p+1)} - x_{-1}) \cdots (x_{-(n+p)} - x_{-n})$. Note that this pre-period can be seen as a difference between two finite expansions and so the normalization will not interfere with the period.

Now let $x \geq 1$, $x \in \mathbb{Q}(\beta)$. Indeed, there exists a positive integer N such that $x < \beta^N$. Hence $t = 1 - \frac{x}{\beta^N} \in \mathbb{Q}(\beta) \cap [0, 1)$. As we have proved before the number $-t' = \frac{x'}{\alpha^N} - 1$ has an eventually periodic α -adic expansion. Therefore an eventually periodic α -adic expansion ${}_{\alpha}\langle x' \rangle$ is simply obtained by adding 1 to ${}_{\alpha}\langle \frac{x'}{\alpha^N} - 1 \rangle$, followed by shifting the fractional point N positions to the left. \square

4 Expansions in bases satisfying Finiteness Property (F)

In the previous section we proved a general result characterizing α -adic expansions of elements of the extension field $\mathbb{Q}(\alpha)$. If we add one additional condition on β , namely that it fulfills Property (F), we are able to characterize the expansions of elements of the ring $\mathbb{Z}[\alpha^{-1}]$ more precisely.

Proposition 14 *Let α be a conjugate of a Pisot number β satisfying Property (F). For any $x \in \mathbb{Z}[\beta^{-1}]_+$ its conjugate x' has at least one α -adic expansion. This expansion is finite and ${}_{\alpha}\langle x' \rangle = \langle x \rangle_{\beta}$.*

PROOF. Since β satisfies Property (F), $\text{Fin}(\beta) = \mathbb{Z}[\beta^{-1}]$ and any $x \in \mathbb{Z}[\beta^{-1}]_+$ has a finite β -expansion, say $x = \sum_{i=-j}^k x_i \beta^i$. By conjugation we have $x' = \sum_{i=-j}^k x_i \alpha^i$. \square

The proof of Proposition 14 shows us a way how to compute an α -adic expansion of a number x' which is a conjugate of $x \in \mathbb{Z}[\beta^{-1}]_+$. The same task is a little bit more complicated in the case where x' is a conjugate of an $x \in \mathbb{Z}[\beta^{-1}]_-$. An α -adic expansion of such a negative number x' is computed by Algorithm 1 below.

Algorithm 1 *Let $x \in \mathbb{Z}[\beta^{-1}]_-$. An α -adic expansion of x' is obtained as follows.*

- (1) *Use the greedy algorithm to find the β -expansion of $-x$, say $\langle -x \rangle_{\beta} = x_k \cdots x_0 \bullet x_{-1} \cdots x_{-j}$, which is finite since β satisfies Property (F).*
- (2) *By changing the signs $x_i \mapsto -x_i$ we obtain an α -adic representation of x' in the form of a finite word over the alphabet $\{0, -1, \dots, -\lfloor \beta \rfloor\}$.*
- (3) *Subtract -1 from the rightmost non-zero coefficient x_{-j} and replace it by an α -adic expansion of -1 , ${}_{\alpha}\langle -1 \rangle = {}^{\omega}(t_1 \cdots t_{\ell-1}(t_{\ell} - 1))$. The representation of x' has now a periodic part ${}^{\omega}(t_1 \cdots t_{\ell-1}(t_{\ell} - 1))$ and a pre-period, which is a finite word over the alphabet $\{-\lfloor \beta \rfloor, \dots, \lfloor \beta \rfloor\}$.*
- (4) *Finally, the α -adic expansion of x' is simply obtained by the normalization of the pre-period. Note that the pre-period can be seen as a difference between two finite expansions and so the normalization will not interfere with the period.*

Example 15 *Let $\beta > 1$ be the golden mean, α its conjugate. Recall that for instance ${}_{\alpha}\langle -1 \rangle = {}^{\omega}(10) \bullet$. We compute an α -adic expansion of the number -4 . The β -expansion of 4 is $101 \bullet 01$, so $\bar{1}0\bar{1} \bullet 0\bar{1}$ is an α -adic representation of the number -4 . Now we subtract -1 from the rightmost non-zero coefficient and*

replace it by ${}_{\alpha}\langle -1 \rangle$ as follows

$$\begin{array}{c} \bar{1} 0 \bar{1} \bullet 0 \bar{1} \\ \bullet 1 \\ \hline \omega(1 0) 1 0 1 0 \bullet 1 0 \\ \omega(1 0) 1 \bar{1} 1 \bar{1} \bullet 1 0 \end{array}$$

Since the normalization of the pre-period $1\bar{1}1\bar{1}\bullet 10$ gives $0100\bullet 001$, the expansion of -4 is ${}_{\alpha}\langle -4 \rangle = \omega(10)0100\bullet 001$.

Proposition 16 *Let α be a conjugate of a Pisot number β satisfying Property (F). For any $x \in \mathbb{Z}[\beta^{-1}]_-$, its conjugate x' has at least ℓ different α -adic expansions, which are eventually periodic to the left with the period $\omega(t_1 \cdots t_{\ell-1}(t_{\ell} - 1))$.*

PROOF. First, we show that the number -1 has ℓ different α -adic expansions. Recall that $-1 + \pi_{\beta}(d_{\beta}(1)) = 0$, hence $-\alpha^{\ell} + \alpha^{\ell}\pi_{\alpha}(d_{\beta}(1)) - 1 = -1$. Therefore we have the first expansion

$${}_{\alpha}\langle -1 \rangle = \omega(t_1 \cdots t_{\ell-1}(t_{\ell} - 1)). \quad (2)$$

Now we successively use the equality $-\alpha^j + \alpha^j\pi_{\alpha}(d_{\beta}(1)) - 1 = -1$ for $j = \ell - 1, \dots, 1$ to obtain the other $\ell - 1$ representations. For a given j such that $t_j \neq 0$ this equation is $-\alpha^j + t_1\alpha^{j-1} + \cdots + t_{j-1}\alpha + (t_j - 1) + t_{j+1}\alpha^{-1} + \cdots + t_{\ell}\alpha^{j-\ell} = -1$. If we replace the coefficient -1 at α^j by its expansion (2) we have

$${}_{\alpha}\langle -1 \rangle = \omega(t_1 \cdots t_{\ell-1}(t_{\ell} - 1))t_1 \cdots t_{j-1}(t_j - 1)\bullet t_{j+1} \cdots t_{\ell}. \quad (3)$$

Note that periods of expansions obtained in (3) are mutually shifted, they are situated on all possible ℓ positions. That is why all these expansions are essentially distinct.

The only difficulty would arise if $t_j = 0$ for some j and hence we would obtain the coefficient -1 at α^0 by equation (3). If this is the case we take the pre-period and normalize it

$$t_1 \cdots t_{j-1}(t_j - 1)\bullet t_{j+1} \cdots t_{\ell} \xrightarrow{\nu_C} u_1 \cdots u_j \bullet u_{j+1} \cdots u_i,$$

where $C = \{-1, 0, \dots, \lfloor \beta \rfloor\}$.

An α -adic expansion of -1 then will be ${}_{\alpha}\langle -1 \rangle = \omega(t_1 \cdots t_{\ell-1}(t_{\ell} - 1))u_1 \cdots u_j \bullet u_{j+1} \cdots u_i$.

Then we consider an $x \in \mathbb{Z}[\beta^{-1}]_-$. Using the ℓ different expansions of -1 in Algorithm 1 gives us ℓ different α -adic expansions of the number x' . \square

Note that, conversely, if an expansion of a number z' is of the form ${}^\omega(t_1 \cdots t_{\ell-1}(t_\ell - 1))u \bullet v$, then z belongs to $\mathbb{Z}[\beta^{-1}]_-$.

Example 17 Let β of minimal polynomial $x^3 - x^2 - 1$; such a number is Pisot and satisfies the (F) property [2]. We have $d_\beta(1) = 101$ and $d_\beta^*(1) = (100)^\omega$. Let α be one of its (complex) conjugates. The number -1 has three different α -adic expansions

$$\begin{aligned} \alpha \langle -1 \rangle &= {}^\omega(100) \bullet \\ \alpha \langle -1 \rangle &= {}^\omega(100)0 \bullet 01 \\ \alpha \langle -1 \rangle &= {}^\omega(100)01 \bullet 00001 \end{aligned}$$

5 Quadratic Pisot units

This final section is devoted to quadratic Pisot units satisfying Property (F), i.e. to numbers β with minimal polynomial of the form $X^2 - aX - 1$, $a \in \mathbb{Z}_+$. Then $\alpha = -\beta^{-1}$. The Rényi expansion of 1 is $d_\beta(1) = a1$, and the canonical alphabet is $A = \{0, \dots, a\}$.

5.1 Unicity of expansions of elements of $\mathbb{Z}[\beta]$

We first establish a technical result.

Proposition 18 Let $\beta > 1$ satisfying the equation $X^2 - aX - 1$, and let α be its algebraic conjugate. Let $\alpha \#(x) : \mathbb{R} \rightarrow \mathbb{N}$ be the function counting the number of different α -adic expansions of a number x in \mathbb{R} . Then $\alpha \#(x) < +\infty$ for any $x \in \mathbb{R}$.

PROOF. Let x be fixed in \mathbb{R} and let $\alpha \langle x \rangle = u \bullet v$ be an α -adic expansion of x , with $u \in {}^\mathbb{N}A$ and $v \in A^*$.

Clearly

$$|x - \pi_\alpha(\bullet v)| = |\pi_\alpha(u \bullet)| < \frac{[\beta]}{1 - |\alpha|}. \quad (4)$$

Since β is a unit, $\pi_\beta(\bullet v)$ belongs to $\mathbb{Z}[\beta] \cap [0, 1)$. So $\pi_\beta(\bullet v)$ is a polynomial in β that takes bounded values as well as its conjugate (according to (4)), hence $\pi_\beta(\bullet v)$ takes a finite number of values. So there is a finite number, say B , of values $\pi_\alpha(\bullet v)$. Moreover the bound B is uniform.

Now suppose that there is a number $y \in \mathbb{R}$ such that y has an infinite number of α -adic expansions. Then there exists a constant N such that $\alpha^{-N}y$ has $B+1$ different fractional parts, which is impossible. \square

Note that the proof does not use the fact that β is quadratic, but only that it is a Pisot unit.

In the case that β is a cubic Pisot unit with complex conjugates satisfying Property (F), Sadahiro has proved in [24] that the number of different α -adic expansions of a given complex number is finite.

Proposition 19 *Let $\beta > 1$ satisfying the equation $X^2 - aX - 1$. Let $x \in \mathbb{Z}[\beta]_+$. Then x' has a unique α -adic expansion. This expansion is finite and such that $\alpha \langle x' \rangle = \langle x \rangle_\beta$.*

PROOF. By Proposition 14 any number $x' \in \mathbb{Z}[\alpha]_+$ has an expansion $\alpha \langle x' \rangle = x_k \cdots x_0 \bullet x_{-1} \cdots x_{-j}$. Let us suppose that x' has another α -adic expansion $\alpha \langle x' \rangle = \cdots u_n \cdots u_0 \bullet u_{-1} \cdots u_{-m}$. Subtracting these two expansions of x' and normalizing the result we obtain an admissible expansion of zero of the form $\cdots u_{k+3} u_{k+2} v_{k+1} \cdots v_0 \bullet v_{-1} \cdots v_{-p}$, with $v_{-p} \neq 0$. By shifting and relabeling

$$0 = \sum_{i \geq 0} \alpha^i z_i, \quad (5)$$

where $(z_i)_{i \geq 0}$ is an admissible sequence with $z_0 \neq 0$. The admissibility condition in this case implies $z_1 \in \{0, \dots, a-1\}$. Since $\alpha = -\beta^{-1}$ one can rewrite (5) as

$$\underbrace{z_0 + \frac{z_2}{\beta^2} + \frac{z_4}{\beta^4} + \cdots}_{=:LS} = \underbrace{\frac{z_1}{\beta} + \frac{z_3}{\beta^3} + \frac{z_5}{\beta^5} + \cdots}_{=:RS}. \quad (6)$$

The coefficients z_i for $i \geq 1$ belong to $\{0, \dots, a\}$, hence by summing the geometric series on both sides of (6) we obtain $LS \in [1, a + \frac{1}{\beta}]$ and $RS \in [0, 1 - \frac{1}{\beta}]$ which is absurd. \square

To prove an analogue of Proposition 19 stating the unicity of α -adic expansions for the elements of $\mathbb{Z}[\beta]_-$ we first need the following lemma.

Lemma 20 *If a number z has an eventually periodic α -adic expansion then all its α -adic expansions are eventually periodic.*

PROOF. From Proposition 19, if a number x has a finite α -adic expansion then this expansion is unique.

Let us consider a number x' with an eventually periodic expansion

$$\alpha\langle x' \rangle = {}^\omega(x_{k+p} \cdots x_{k+1})x_k \cdots x_0 \bullet x_{-1} \cdots x_{-j}. \quad (7)$$

For the sake of contradiction let us assume that x' has another α -adic expansion, which is infinite and non-periodic

$$\alpha\langle x' \rangle = \cdots u_1 u_0 \bullet u_{-1} \cdots u_{-m}. \quad (8)$$

Put $y' := \alpha^{-(k+1)}x' - \pi_\alpha(0 \bullet x_k \cdots x_0 x_{-1} \cdots x_{-j})$. Hence from (7) we have

$$\alpha\langle y' \rangle = {}^\omega(x_{k+p} \cdots x_{k+1}) \bullet 0. \quad (9)$$

From (8), defining $v_{k+1} \bullet v_k \cdots v_0 v_{-1} \cdots v_{-q}$ as the word obtained by normalization of the result of digit-wise subtraction $u_{k+1} \bullet u_k \cdots u_1 u_0 u_{-1} \cdots u_{-m} - 0 \bullet x_k \cdots x_0 x_{-1} \cdots x_{-j}$, we have

$$\alpha\langle y' \rangle = \cdots u_{k+3} u_{k+2} v_{k+1} \bullet v_k \cdots v_0 v_{-1} \cdots v_{-q}, \quad (10)$$

which is non-periodic.

Equation (9) gives us another formula for y' , $y' = \alpha^{-p}y' - \pi_\alpha(0 \bullet x_{k+p} \cdots x_{k+1})$. Iterating this formula on the non-periodic expansion (10) yields infinitely many different α -adic expansions of the number y' . This is in contradiction with the statement of Proposition 18. \square

Proposition 21 *Let $\beta > 1$ satisfying the equation $X^2 - aX - 1$. Let $x \in \mathbb{Z}[\beta]_-$. Then x' has exactly two eventually periodic α -adic expansions with period ${}^\omega(a0)$.*

PROOF. At first, we prove that the number -1 has no other α -adic expansions than those from Proposition 16. Since all α -adic expansions of -1 have to be eventually periodic, we will discuss only two cases: when the period is ${}^\omega(a0)$ and when it is different.

(1) Consider an α -adic expansion of -1 with period ${}^\omega(a0)$

$$\begin{aligned} \alpha\langle -1 \rangle &= {}^\omega(a0)d_k \cdots d_0 \bullet d_{-1} \cdots d_{-j}, \\ -1 &= -\alpha^{k+1} + \pi_\alpha(d_k \cdots d_0 \bullet d_{-1} \cdots d_{-j}). \end{aligned}$$

The number $-1 + \alpha^{k+1}$ is the conjugate of $\beta^{k+1} - 1 \in \mathbb{Z}[\beta]_+$ and as such has a unique α -adic expansion. Therefore there cannot be two different pre-periods for a given position of the period.

(2) Suppose that -1 has an α -adic expansion with a different period

$$\alpha\langle -1 \rangle = {}^\omega(d_{k+p} \cdots d_{k+1})d_k \cdots d_0 \bullet d_{-1} \cdots d_{-j}.$$

Let $P' := \pi_\alpha(d_{k+p} \cdots d_{k+1})$. Then

$$-1 = \alpha^{k+1} \frac{P'}{1 - \alpha^p} + \pi_\alpha(d_k \cdots d_0 \bullet d_{-1} \cdots d_{-j}),$$

and by taking the conjugate we obtain

$$-1 = \beta^{k+1} \frac{P}{1 - \beta^p} + \pi_\beta(d_k \cdots d_0 \bullet d_{-1} \cdots d_{-j}).$$

Therefore

$$\underbrace{\pi_\beta(d_k \cdots d_0 \bullet d_{-1} \cdots d_{-j}) + 1}_{\in \mathbb{Z}[\beta]_+} = \underbrace{\beta^{k+1} \frac{P}{\beta^p - 1}}_{\notin \mathbb{Z}[\beta]_+},$$

which is a contradiction.

Validity of the statement for numbers $x' \in \mathbb{Z}[\alpha]_-$, $x' \neq -1$, is then a simple consequence of Algorithm 1. \square

5.2 Representations of rational numbers

In this subsection we inspect α -adic expansions of rational numbers for α the conjugate of a quadratic Pisot unit satisfying (F). We give below an algorithm for computing an α -adic representation of a rational number $q \in \mathbb{Q}$, $|q| < 1$. The algorithm for computing ${}_\alpha\langle q \rangle$ is a sort of a right to left normalization — it consists of successive transformations of a representation of q , and it gives as a result a left infinite sequence on the canonical alphabet A .

Let x_1, x_2 and x_3 be rational numbers, and define the following transformation

$$\psi : (x_3, x_2, x_1) \mapsto (x_3 - (\lceil x_1 \rceil - x_1), x_2 + a(\lceil x_1 \rceil - x_1), \lceil x_1 \rceil). \quad (11)$$

Note that this transformation preserves the α -value.

Algorithm 2 *Input:* $q \in \mathbb{Q} \cap (-1, 1)$.

Output: a sequence $s = (s_i)_{i \geq 0}$ of $A^{\mathbb{N}}$ such that $\sum_{i \geq 0} s_i \alpha^i = q$.

begin

$s_0 := q;$

for $i \geq 1$ **do** $s_i := 0;$

$i := 0;$

repeat

$(s_{i+2}, s_{i+1}, s_i) := \psi(s_{i+2}, s_{i+1}, s_i);$

$i := i + 1;$

end

Since the starting point of the whole process is a single rational number, after each step there will be at most two non-integer coefficients — rational numbers with the same denominator as q .

Denote $s^{(i+1)}$ the resulting sequence after step i ; thus $s^{(0)} = \omega 0q$ and, for $i \geq 0$, $s^{(i+1)} = \dots s_{i+4}^{(i+1)} s_{i+3}^{(i+1)} s_{i+2}^{(i+1)} s_{i+1}^{(i+1)} s_i^{(i+1)} \dots s_0^{(i+1)}$ where the digits $s_0^{(i+1)} = s_0, \dots, s_i^{(i+1)} = s_i$ are integer digits of the output, and the factor $s_{i+3}^{(i+1)} s_{i+2}^{(i+1)} s_{i+1}^{(i+1)}$ is under consideration. Note that for $j \geq i+3$, the coefficients $s_j^{(i+1)}$ are all equal to 0. Thus the next iteration of the algorithm gives $\psi(s_{i+3}^{(i+1)}, s_{i+2}^{(i+1)}, s_{i+1}^{(i+1)}) = (s_{i+3}^{(i+2)}, s_{i+2}^{(i+2)}, s_{i+1}^{(i+2)})$.

Lemma 22 *After every step i of the algorithm, the coefficients satisfy:*

- $s_0^{(i+1)} = s_0, \dots, s_i^{(i+1)} = s_i$ belong to A
- $s_{i+1}^{(i+1)} \in (-1, a)$
- $s_{i+2}^{(i+1)} \in (-1, 0]$.

PROOF. We will prove the statement by induction on the number of steps of the algorithm. The statement is valid for $i = 0$ due to the assumption $|q| < 1$.

By Transformation (11) we have $\psi(s_{i+3}^{(i+1)}, s_{i+2}^{(i+1)}, s_{i+1}^{(i+1)}) = (s_{i+3}^{(i+2)}, s_{i+2}^{(i+2)}, s_{i+1}^{(i+2)})$, thus

$$\begin{aligned} s_{i+1}^{(i+2)} &= \lceil s_{i+1}^{(i+1)} \rceil \in \mathbb{Z} \cap [0, a], \\ s_{i+2}^{(i+2)} &= s_{i+2}^{(i+1)} + a(\lceil s_{i+1}^{(i+1)} \rceil - s_{i+1}^{(i+1)}) \in (-1, a), \\ s_{i+3}^{(i+2)} &= -(\lceil s_{i+1}^{(i+1)} \rceil - s_{i+1}^{(i+1)}) \in (-1, 0]. \end{aligned}$$

□

Since the factor $s_{i+3}^{(i+2)} s_{i+2}^{(i+2)} s_{i+1}^{(i+2)}$ after step $i+1$ is uniquely determined by the factor $s_{i+3}^{(i+1)} s_{i+2}^{(i+1)} s_{i+1}^{(i+1)}$, and the coefficients $s_{i+1}^{(i+1)}$ and $s_{i+2}^{(i+1)}$ are uniformly bounded, as a corollary we get the following result.

Proposition 23 *Algorithm 2 generates an α -adic representation on A of q which is eventually periodic.*

Note that in general, this representation is not admissible.

Example 24 *Computation of an α -adic representation of the number $\frac{1}{2}$ in*

the case $d_\beta(1) = 31$.

$$\begin{array}{ccccccc}
 & & & & & & \frac{1}{2} \\
 & & & & & -\frac{1}{2} & \frac{3}{2} & \frac{1}{2} \\
 \hline
 & & & & & -\frac{1}{2} & \frac{3}{2} & 1 \\
 & & & & & -\frac{1}{2} & \frac{3}{2} & \frac{1}{2} \\
 \hline
 & & & & & -\frac{1}{2} & 1 & 2 & 1 \\
 & & & & & -\frac{1}{2} & \frac{3}{2} & \frac{1}{2} \\
 \hline
 & & & & & -\frac{1}{2} & \frac{3}{2} & 0 & 1 & 2 & 1
 \end{array}$$

Because the prefix $(0)(-\frac{1}{2})(\frac{3}{2})$ which arises after step 3 is the same as the one which arises after step 0, the same sequence of steps (with the same results) will follow from now on. Therefore the α -adic representation computed by the algorithm is ${}_\alpha\langle\frac{1}{2}\rangle = \omega(012)1\bullet$. It happens that, in this particular case, this is an α -adic expansion of ${}_\alpha\langle\frac{1}{2}\rangle$.

5.3 Normalization

Unfortunately, Algorithm 2 does not give in general an admissible α -adic expansion, that is to say a representation with no factor $a1$. In this section we discuss the normalization of such a non-admissible output. Since the output word is a word on the canonical alphabet, non-admissibility can come from a factor of the type ab with $b > 0$, or $a^n b$ with $n > 1$, $b \neq a$, or from an infinite prefix of the type ${}^\omega a$. The following result shows that the latter case will not appear.

Proposition 25 *The number of consecutive letters a 's in an output word of Algorithm 2 is bounded for all $q \in \mathbb{Q} \cap (-1, 1)$.*

PROOF. We will prove the result by contradiction. Let us assume that from some step on, say from step i , the output of the algorithm is composed only of letters a 's. This means that the output is of the form $\cdots [V_4][V_3][V_2][V_1]v$, where v has length $i + 1$, and for each $k \geq 1$, $[V_k] = a$. We have $V_1 = s_{i+1}^{(i+1)}$, and $V_2 = s_{i+2}^{(i+1)} + a([V_1] - V_1)$. Iterating twice the transformation ψ , we get

$$V_k = -([V_{k-2}] - V_{k-2}) + a([V_{k-1}] - V_{k-1}) \quad \text{for } k \geq 3. \quad (12)$$

From (12) and the fact that $V_k > a - 1$ one gets

$$1 - \frac{1}{a} + \frac{1}{a}([V_{k-2}] - V_{k-2}) < ([V_{k-1}] - V_{k-1}). \quad (13)$$

Then iterating (13) we obtain an explicit estimate for $(\lceil V_k \rceil - V_k)$

$$\begin{aligned}
(\lceil V_k \rceil - V_k) &> 1 - \frac{1}{a} + \frac{1}{a}(\lceil V_{k-1} \rceil - V_{k-1}) \\
&> 1 - \frac{1}{a} + \frac{1}{a} \left(1 - \frac{1}{a} + \frac{1}{a}(\lceil V_{k-2} \rceil - V_{k-2}) \right) \\
&= 1 - \frac{1}{a^2} + \frac{1}{a^2}(\lceil V_{k-2} \rceil - V_{k-2}) \\
&> 1 - \frac{1}{a^3} + \frac{1}{a^3}(\lceil V_{k-3} \rceil - V_{k-3}) \\
&> \dots \\
&> 1 - \frac{1}{a^{k-1}} + \frac{1}{a^{k-1}}(\lceil V_1 \rceil - V_1)
\end{aligned}$$

Since $s_{i+2}^{(i+1)} \in (-1, 0]$ we can estimate $a - 1 < V_2 = s_{i+2}^{(i+1)} + a(\lceil V_1 \rceil - V_1) \leq a(\lceil V_1 \rceil - V_1)$, which gives $1 - \frac{1}{a} < (\lceil V_1 \rceil - V_1)$. Therefore we have

$$-(\lceil V_k \rceil - V_k) < -1 + \frac{1}{a^{k-1}} - \frac{1}{a^{k-1}} \left(1 - \frac{1}{a} \right) = \frac{1}{a^k} - 1. \quad (14)$$

Finally, by inequality (14) and the fact that $a - 1 < V_k \leq a$, we obtain a bound on V_k

$$V_k = \underbrace{-(\lceil V_{k-2} \rceil - V_{k-2})}_{< \frac{1}{a^{k-2}} - 1} + \underbrace{a\lceil V_{k-1} \rceil}_{=a^2} - \underbrace{aV_{k-1}}_{< a-a^2} < a - 1 + \frac{1}{a^{k-2}}. \quad (15)$$

Suppose that we are computing an α -adic expansion of a rational number q with denominator $p \in \mathbb{N}$. Find the smallest K such that $\frac{1}{p} > \frac{1}{a^{K-2}}$. Since any V_k is a fraction with denominator p of the form $V_K = \frac{t}{p}$, by (15) we have $V_K < a - 1 + \frac{1}{a^{K-2}}$, which implies $V_K < a - 1$. This is in contradiction with the assumption that $a - 1 < V_k$ for all $k \geq 1$. \square

Proposition 26 *Let w be an output of Algorithm 2 for a number $q \in \mathbb{Q} \cap (-1, 1)$ and let \hat{w} be the image of w under the normalization function, $\nu_A(w) = \hat{w}$. Then \hat{w} is left eventually periodic with no fractional part.*

PROOF. First of all, let us recall that a number β such that $d_\beta(1) = a1$ is a so-called *confluent Pisot number* (see [11]). For these numbers, it is known that the normalization on the canonical alphabet does not produce a carry to the right. This assures that \hat{w} will have no fractional part and that we can perform normalization starting from the fractional point and then just read and write from right to left.

We have shown earlier that for a given rational number q the number of consecutive letters a 's in an output word w is bounded, moreover the proof

of Proposition 25 gives us this upper bound. We give below a construction of a right sequential finite transducer \mathcal{T} performing the normalization of such a word w .

Define $A_\emptyset := A \setminus \{0\}$, and let C be the bound on the number of consecutive letters a in a word w . Because the result of the normalization of non-admissible factors of w depends on the parity of the length of blocks of consecutive a 's, the transducer \mathcal{T} has to count this parity. This is done by memorizing the actually processed forbidden factors; the states of the transducer are labeled by these memorized factors.

Transducer \mathcal{T} is constructed as follows

- The initial state is labeled by the empty word ε , and there is a loop $\varepsilon \xrightarrow{0|0} \varepsilon$.
- There are states labeled by a single letter $h \in A_\emptyset$ connected with the initial state by edges $\varepsilon \xrightarrow{h|\varepsilon} h$ and $h \xrightarrow{0|0h} \varepsilon$. These states are also connected one with each other by edges $i \xrightarrow{j|i} j$ where $i, j \in A_\emptyset$, $j \neq a$. Finally there is a loop $h \xrightarrow{h|h} h$ on each state $h \in A_\emptyset$, $h \neq a$.
- For each $h \in A_\emptyset$ there is a chain of consecutive states $a^k h$, where $k = 1, \dots, C - 1$, linked by edges $a^k h \xrightarrow{a|\varepsilon} a^{k+1} h$. Moreover, there are edges $a^k h \xrightarrow{i|u} i+1$ where $u = (0a)^m 0(h-1)$ for $k = 2m+1$ and $u = (0a)^m 0(a-1)h$ for $k = 2m+2$.

The edges $a^k h \xrightarrow{a|\varepsilon} a^{k+1} h$ are these which count the number of consecutive letters a in a forbidden factor, whereas the edges $a^k h \xrightarrow{i|u} i+1$ are these which, depending on the parity of the length k of a run a^k , replace a forbidden factor by its normalized equivalent.

One can easily check that the transducer is input deterministic, and thus right sequential. Clearly the output word is admissible. Since the image by a sequential function of an eventually periodic word is eventually periodic (see [10]), the image \hat{w} is eventually periodic. \square

The following is just a rephrasing.

Theorem 27 *Let β be a quadratic Pisot unit satisfying (F). Any rational number $q \in \mathbb{Q} \cap (-1, 1)$ has an eventually periodic α -adic expansion with no fractional part.*

Remark that there exist rational numbers larger than 1 such that the α -adic expansion has no fractional part. We have shown in Example 24 that for $d_\beta(1) = 31$, ${}_\alpha\langle \frac{1}{2} \rangle = {}^\omega(012)1\bullet$. Thus ${}_\alpha\langle \frac{3}{2} \rangle = {}^\omega(012)2\bullet$ has no fractional part.

6 Conclusion

Let us stress out that the analogue of Propositions 19 and 21 has been proved by Sadahiro for the case that β is a cubic Pisot unit with complex conjugates satisfying Property (F). The extension of these results to other Pisot units satisfying Property (F) is an open problem.

We have not addressed here the question of the existence of α -adic expansions of complex numbers not in $\mathbb{Q}(\alpha)$.

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