Matrices of 3iet preserving morphisms

P. Ambrož   Z. Masáková   E. Pelantová

Doppler Institute & Department of Mathematics
FNSPE, Czech Technical University, Trojanova 13, 120 00 Praha 2, Czech Republic
E-mail: petr.ambroz@fjfi.cvut.cz, masakova@km1.fjfi.cvut.cz,
pelantova@km1.fjfi.cvut.cz

Abstract

We study matrices of morphisms preserving the family of words coding 3-interval exchange transformations. It is well known that matrices of morphisms preserving sturmian words (i.e. words coding 2-interval exchange transformations with the maximal possible factor complexity) form the monoid $\{\boldsymbol{M} \in \mathbb{N}^{2\times 2} \mid \det \boldsymbol{M} = \pm 1\} = \{\boldsymbol{M} \in \mathbb{N}^{2\times 2} \mid \boldsymbol{MEM}^T = \pm \boldsymbol{E}\}$, where $\boldsymbol{E} = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$.

We prove that in case of exchange of three intervals, the matrices preserving words coding these transformations and having the maximal possible subword complexity belong to the monoid $\{\boldsymbol{M} \in \mathbb{N}^{3\times 3} \mid \boldsymbol{MEM}^T = \pm \boldsymbol{E},\ \det \boldsymbol{M} = \pm 1\}$, where $\boldsymbol{E} = \left(\begin{smallmatrix} 0 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & -1 & 0 \end{smallmatrix}\right)$.

1 Introduction

Sturmian words are the most studied class of infinite aperiodic words. By their nature, they are defined purely over a binary alphabet. There exist several equivalent definitions of sturmian words [5], which give rise to several different generalizations of sturmian words over larger alphabets. For example, the generalization of sturmian words to Arnoux-Rauzy words of order r uses the characterization of sturmian words by means of the so-called left and right special factors [2].

Another natural generalization can be derived from the definition of a sturmian word as an aperiodic word coding a transformation of exchange of two intervals. The r-interval exchange transformation has been introduced by Katok and Stepin [14]: An exchange T of r intervals is defined by a vector of r lengths and by a permutation of r letters; the unit interval is then partitioned according to the vector of lengths, and T interchanges these intervals according to the given permutation. Rauzy was the first one to observe that interval exchange transformation can be used for the generalization of sturmian words.

In contrast to ergodic properties of these transformations, which were studied by many authors [15, 20, 22, 23], combinatorial properties of associated words have been so far explored only a little. Some results, analogical to the properties known for sturmian words, have been derived for the most simple case, namely for 3-interval exchange transformations. Note that for the exchange of three intervals, the most interesting permutation is (321) and all the results cited below apply to transformations with this permutation. Words coding 3-interval exchange transformation can be periodic or aperiodic, depending on the choice of parameters. In accordance with the terminology introduced by [9], infinite words which code 3-interval exchange transformations and are aperiodic, are called 3iet words. The factor complexity $\mathcal{C}_u(n)$ of a 3iet word u, i.e., the number of different factors of length n occurring in u, is known to satisfy $\mathcal{C}_u(n) \leq 2n+1$ for all $n \in \mathbb{N}$. Words for which $\mathcal{C}_u(n) = 2n+1$, for all $n \in \mathbb{N}$, are called non-degenerated (or regular) 3iet words.

In the paper [10], minimal sequences coding 3-interval exchange transformations are fully characterized. The structure of palindromes of these words was described in [9, 4], whereas the paper [10] deals with their return words. Here we study morphisms which map the set of 3iet words to itself.

Morphisms preserving sturmian words were completely described by Berstel, Mignosi and Séébold [6, 17, 21]. Recall that there are two ways to define such a morphism:

- A morphism φ over the binary alphabet $\{0,1\}$ is said to be *locally sturmian* if there is a sturmian word u such that $\varphi(u)$ is also sturmian.
- A morphism φ over the binary alphabet $\{0,1\}$ is said to be *sturmian* if $\varphi(u)$ is sturmian for all sturmian words u.

Berstel, Mignosi and Séébold showed that the families of sturmian and locally sturmian morphisms coincide and that they form a monoid generated by three morphisms, ψ_1, ψ_2 and ψ_3, given by

$$\psi_1 : \begin{matrix} 0 \mapsto 01 \\ 1 \mapsto 1 \end{matrix}, \qquad \psi_2 : \begin{matrix} 0 \mapsto 10 \\ 1 \mapsto 1 \end{matrix}, \qquad \psi_3 : \begin{matrix} 0 \mapsto 1 \\ 1 \mapsto 0 \end{matrix}. \tag{1}$$

To each morphism φ over a k-letter alphabet $\{a_1, \ldots, a_k\}$ one can assign its incidence matrix $\boldsymbol{M}_\varphi \in \mathbb{N}^{k\times k}$ by putting

$$(\boldsymbol{M}_\varphi)_{ij} = \text{number of letters } a_j \text{ in the word } \varphi(a_i)\,. \tag{2}$$

As a simple consequence of the fact that the monoid of sturmian morphisms is generated by ψ_1, ψ_2 and ψ_3 from (1), one has the following fact: *A matrix $\boldsymbol{M} \in \mathbb{N}^{2\times 2}$ is the incidence matrix of a sturmian morphism if and only if* $\det \boldsymbol{M} = \pm 1$. By an easy calculation we can derive that for matrices of order 2×2

$$\det \boldsymbol{M} = \pm 1 \quad \Longleftrightarrow \quad \boldsymbol{M}\boldsymbol{E}\boldsymbol{M}^T = \pm \boldsymbol{E}\,, \text{ where } \boldsymbol{E} = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right).$$

In the theory of Lie groups, one can formulate this claim by stating that the group $\mathrm{SL}(2,\mathbb{Z})$ is isomorphic to the group $\mathrm{Sp}(2,\mathbb{Z})$, see [13].

The aim of this paper is to derive similar properties for matrices of morphisms preserving the family of 3iet words, which we call here 3iet preserving morphisms. We will prove the following theorems.

Theorem A. *Let φ be a 3iet preserving morphism and let $\boldsymbol{M}$ be its incidence matrix. Then*

$$\boldsymbol{M}\boldsymbol{E}\boldsymbol{M}^T = \pm \boldsymbol{E}, \text{ where } \boldsymbol{E} = \left(\begin{smallmatrix} 0 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & -1 & 0 \end{smallmatrix}\right).$$

Theorem B. *Let φ be a 3iet preserving morphism and let $\boldsymbol{M}$ be its incidence matrix. Then one of the following holds*

- $\det \boldsymbol{M} = \pm 1$ *and $\varphi(u)$ is non-degenerated for every non-degenerated 3iet word u,*
- $\det \boldsymbol{M} = 0$ *and $\varphi(u)$ is degenerated for every 3iet word u.*

In the proof of Theorem A we use the description of matrices of sturmian morphisms given above, while the main tool employed in the proof of Theorem B is the connection between words coding 3-interval exchange transformations and cut-and-project sets.

Note that we do not address at all the description of the 3iet preserving morphisms themselves.

2 Preliminaries

In this paper we deal with finite and infinite words over a finite alphabet $\mathcal{A}$, whose elements are called letters. The set of all finite words over $\mathcal{A}$ is denoted by $\mathcal{A}^*$. This set, equipped with the concatenation as a binary operation, is a free monoid having the empty word as its identity. The length of a word $w = w_1 w_2 \cdots w_n$ is denoted by $|w| = n$, the number of letters a_i in the word w is denoted by $|w|_{a_i}$.

2.1 Infinite words

The set of two-sided infinite words over an alphabet $\mathcal{A}$, i.e., of two-sided infinite sequences of letters of $\mathcal{A}$, is denoted by $\mathcal{A}^{\mathbb{Z}}$, its elements are words $u = (u_n)_{n\in\mathbb{Z}}$. Note that in all our considerations we will not identify infinite words $(u_{n+k})_{n\in\mathbb{Z}}$ and $(u_n)_{n\in\mathbb{Z}}$, and therefore we will mark the position corresponding to the index 0, usually using | as the delimiter, e.g. for $u \in \mathcal{A}^{\mathbb{Z}}$,

$$u = \cdots u_{-3}u_{-2}u_{-1}|\, u_0 u_1 u_2 \cdots .$$

The words of this form are sometimes called pointed biinfinite words. Naturally, one can define a metric on the set $\mathcal{A}^{\mathbb{Z}}$.

Definition. Let $u = (u_n)_{n\in\mathbb{Z}}$ and $v = (v_n)_{n\in\mathbb{Z}}$ be two biinfinite words over $\mathcal{A}$. We define the *distance* $\mathrm{d}(u, v)$ between u and v by setting

$$\mathrm{d}(u, v) := \frac{1}{1 + j}\,, \tag{3}$$

where $j \in \mathbb{N}$ is the minimal index such that either $u_j \neq v_j$ or $u_{-j} \neq v_{-j}$.

It can be easily verified that the above defined distance $\mathrm{d}(u, v)$ is a metric and that the set $\mathcal{A}^{\mathbb{Z}}$ with this metric is a compact metric space.

We consider also one-sided infinite words $u = (u_n)_{n\in\mathbb{N}}$, either right-sided $u = u_0 u_1 u_2 \cdots$ or left-sided $u = \cdots u_2 u_1 u_0$.

The degree of diversity of an infinite word u is expressed by the complexity function, which counts the number of factors of length n in the word u. Formally, a word w of length n is said to be a *factor* of a word $u = (u_n)_{n\in\mathbb{Z}}$ if there is an index $i \in \mathbb{Z}$ such that $w = u_i u_{i+1} \cdots u_{i+n-1}$. The set of all factors of u of length n is denoted by $\mathcal{L}_n(u)$. The *language* $\mathcal{L}(u)$ of an infinite word u is the set of all its factors, that is,

$$\mathcal{L}(u) = \bigcup_{n\in\mathbb{N}} \mathcal{L}_n(u)\,.$$

The *(factor) complexity* $\mathcal{C}_u$ of an infinite word u is the function $\mathcal{C}_u : \mathbb{N} \to \mathbb{N}$ defined as

$$\mathcal{C}_u(n) := \#\mathcal{L}_n(u)\,.$$

Clearly, $\mathcal{C}_u(n)$ is a non-decreasing function. Recall that if there exists $n_0 \in \mathbb{N}$ such that $\mathcal{C}_u(n_0) \leq n_0$, then the word u is eventually periodic (if $u = (u_n)_{n\in\mathbb{N}}$), or periodic (if $u = (u_n)_{n\in\mathbb{Z}}$), see [18]. Hence for an aperiodic word u, one has $\mathcal{C}_u(n) \geq n+1$, for all $n \in \mathbb{N}$.

A one-sided sturmian word $(u_n)_{n\in\mathbb{N}}$ is often defined as an aperiodic word with complexity $C_u(n) = n+1$, for all $n \in \mathbb{N}$. However, for biinfinite words, the condition $\mathcal{C}_u(n) \geq n+1$ is not enough for u to be aperiodic. For example, the word $\cdots 111|000\cdots$ has the complexity $\mathcal{C}(n) = n+1$ for all $n \in \mathbb{N}$. In order to define a biinfinite sturmian word $(u_n)_{n\in\mathbb{Z}}$ by means of complexity, we need to add another condition. We introduce the notion of the density of letters, representing the frequency of occurrence of a given letter in an infinite word.

The *density of a letter* $a \in \mathcal{A}$ in a word $u \in \mathcal{A}^{\mathbb{Z}}$ is defined as

$$\rho(a) := \lim_{n\to\infty} \frac{\#\{i \mid -n \leq i \leq n,\ u_i = a\}}{2n+1}\,,$$

if the limit exists.

A biinfinite word $u = (u_n)_{n\in\mathbb{Z}}$ is called sturmian, if $\mathcal{C}_u(n) = n+1$ for each $n \in \mathbb{N}$ and the densities of letters are irrational.

Another equivalent definition of sturmian words uses the balance property. We say that an infinite word u over the alphabet $\{0,1\}$ is *balanced*, if for every pair of factors $v, w \in \mathcal{L}_n(u)$ we have $\big||v|_0 - |w|_0\big| \leq 1$. A one-sided infinite word over the alphabet $\{0,1\}$ is sturmian, if and only if it is balanced. A biinfinite word over $\{0,1\}$ is sturmian, if and only if it is balanced and has irrational densities of letters. For other properties of one-sided and two-sided infinite sturmian words the reader is referred to [16, 19].

Unlike the metric space $\mathcal{A}^{\mathbb{Z}}$, the set of all sturmian words equipped with the same metric (3) is not compact, however we have the following result.

Lemma 1. *Let $u \in \{0,1\}^{\mathbb{Z}}$ be a limit of a sequence of sturmian words $u^{(m)}$. Then u is either sturmian or the densities of letters in u are rational.*

Proof. Let $w, \widehat{w} \in \mathcal{L}(u)$ be factors of the same length in u. Since $u = \lim_{m\to\infty} u^{(m)}$ there exists $m_0 \in \mathbb{N}$ such that $w, \widehat{w}$ are factors of $u^{(m_0)}$, which is sturmian. Therefore $\big||w|_0 - |\widehat{w}|_0\big| \leq 1$ and u is balanced. If, moreover, the densities are irrational, then u is sturmian. The statement follows. ☐

2.2 Morphisms and incidence matrices

A mapping $\varphi : \mathcal{A}^* \to \mathcal{A}^*$ is said to be a *morphism* over $\mathcal{A}$ if $\varphi(w\widehat{w}) = \varphi(w)\varphi(\widehat{w})$ holds for any pair of finite words $w, \widehat{w} \in \mathcal{A}^*$. Obviously, a morphism is uniquely determined by the images $\varphi(a)$ for all letters $a \in \mathcal{A}$.

The action of a morphism φ can be naturally extended to biinfinite words by the prescription

$$\varphi(u) = \varphi(\cdots u_{-2}u_{-1}|\, u_0u_1\cdots) := \cdots\varphi(u_{-2})\varphi(u_{-1})|\,\varphi(u_0)\varphi(u_1)\cdots\,.$$

The mapping $\varphi : u \mapsto \varphi(u)$ is continuous on $\mathcal{A}^{\mathbb{Z}}$; a word $u \in \mathcal{A}^{\mathbb{Z}}$ is said to be a *fixed point* of φ if $\varphi(u) = u$.

Recall that the incidence matrix of a morphism φ over the alphabet $\mathcal{A}$ is defined by (2). A morphism φ is called primitive if there exist an integer k such that the matrix $\boldsymbol{M}_\varphi^k$ is positive.

Morphisms over $\mathcal{A}$ form a monoid, whose neutral element is the identity morphism. Let φ and ψ be morphisms over $\mathcal{A}$, then the matrix of their composition, that is, of the morphism $u \mapsto (\varphi \circ \psi)(u) = \varphi\big(\psi(u)\big)$ is obtained by

$$\boldsymbol{M}_{\varphi\circ\psi} = \boldsymbol{M}_{\psi}\boldsymbol{M}_{\varphi}\,. \tag{4}$$

Let us now explain the importance of the incidence matrix of a morphism φ for the combinatorial properties of infinite words on which the morphism φ acts. Assume that an infinite word u over the alphabet $\mathcal{A} = \{a_1, \ldots, a_k\}$ has well defined densities of letters, given by the vector

$$\vec{\rho}_u = \big(\rho(a_1), \ldots, \rho(a_k)\big)\,.$$

It is easy to see that the densities of letters in the infinite word $\varphi(u)$ are also well defined and it holds that

$$\vec{\rho}_{\varphi(u)} = \frac{\vec{\rho}_u \boldsymbol{M}_{\varphi}}{\vec{\rho}_u \boldsymbol{M}_{\varphi}\begin{pmatrix}1\\ \vdots\\ 1\end{pmatrix}}\,, \tag{5}$$

where $\boldsymbol{M}_{\varphi}$ is the incidence matrix of φ.

Assume now that the infinite word u is a fixed point of a morphism φ. Then from (5), we obtain that the vector of densities $\vec{\rho}_u$ is a left eigenvector of the incidence $\boldsymbol{M}_{\varphi}$, i.e., $\vec{\rho}_u\boldsymbol{M}_{\varphi} = \Lambda\vec{\rho}_u$. Since $\boldsymbol{M}_{\varphi}$ is a non-negative integral matrix, we can use the Perron-Frobenius Theorem stating that Λ is the dominant eigenvalue of $\boldsymbol{M}_{\varphi}$. Moreover, all eigenvalues of $\boldsymbol{M}_{\varphi}$ are algebraic integers.

The right eigenvector of the incidence matrix corresponding to the dominant eigenvalue has also a nice interpretation. It plays an important role for the *geometric representation* of a fixed point of a morphism. Let u be a fixed point of a morphism φ over a k-letter alphabet $\{a_1, \ldots, a_k\}$ and let $\boldsymbol{M}_{\varphi}$ have a positive right eigenvector $\vec{x}$. The infinite word u can be geometrically represented by a self-similar set Σ as follows.

Let us denote by $x_1, x_2, \ldots, x_k$ the positive components of $\vec{x}$, and let Λ be the corresponding eigenvalue, i.e., $\boldsymbol{M}_{\varphi}\vec{x} = \Lambda\vec{x}$. Since $\boldsymbol{M}_{\varphi}$ is non-negative and $\vec{x}$ is positive, the eigenvalue Λ is equal to the spectral radius of the matrix $\boldsymbol{M}_{\varphi}$. Moreover, $\boldsymbol{M}_{\varphi}$ being an integral matrix implies $\Lambda \geq 1$.

For a biinfinite word $u = \cdots u_{-3}u_{-2}u_{-1}|u_0u_1u_2\cdots$ we denote

$$\Sigma = \Big\{\sum_{i=1}^{k} |w|_{a_i} x_i \;\Big|\; w \text{ is an arbitrary prefix of } u_0u_1u_2\cdots\Big\}$$
$$\cup \Big\{-\sum_{i=1}^{k} |w|_{a_i} x_i \;\Big|\; w \text{ is an arbitrary suffix of } \cdots u_{-3}u_{-2}u_{-1}\Big\}\,.$$

The set Σ can be equivalently defined as

$$\Sigma = \{t_n \mid n \in \mathbb{Z}\}\,, \quad \text{where} \quad t_0 = 0 \;\text{ and }\; t_{n+1} - t_n = x_i \;\Leftrightarrow\; u_n = a_i\,.$$

Since u is a fixed point of a morphism, the construction of Σ implies that $\Lambda\Sigma \subset \Sigma$. A set having this property is called self-similar.

Moreover, if $u_n = a_i$ then the number of points of the set Σ belonging to $(\Lambda t_n, \Lambda t_{n+1}]$ is equal to the length of $\varphi(a_i)$. Formally, we have

$$\#\big((\Lambda t_n, \Lambda t_{n+1}] \cap \Sigma\big) = |\varphi(a_i)|\,. \tag{6}$$

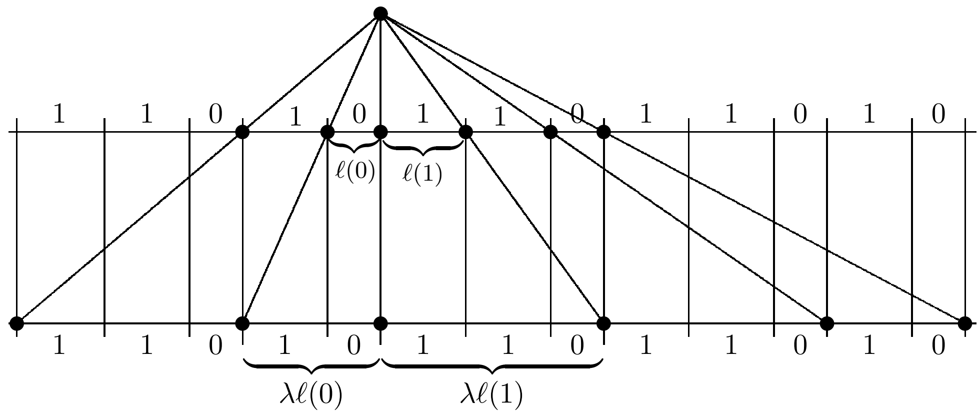

Figure 1: Action of the morphism $0 \mapsto 10$, $1 \mapsto 110$ on the geometric representation of its fixed point $u = \lim_{n\to\infty} \varphi^n(0)|\varphi^n(1)$.

In Figure 1, one can see the geometric representation of the fixed point of the morphism $0 \mapsto 10$, $1 \mapsto 110$. The matrix of this morphism, $\boldsymbol{M} = \left(\begin{smallmatrix} 1 & 1 \\ 1 & 2 \end{smallmatrix}\right)$, has the dominant eigenvalue $\Lambda = \tau^2$, where $\tau = \frac{1+\sqrt{5}}{2}$ is the golden ratio. The corresponding right eigenvector of M is $\left(\begin{smallmatrix} 1 \\ \tau \end{smallmatrix}\right)$. Hence the lengths assigned to letters 0 and 1 are $\ell(0) = 1$ and $\ell(1) = \tau$, respectively.

3 Interval exchange words

Before we define infinite words coding a 3-interval exchange transformation, we will show the definition of sturmian words using a 2-interval exchange transformation. It is well known (see e.g. [18, 16]) that every sturmian word $u = (u_n)_{n\in\mathbb{Z}}$ over the alphabet $\{0,1\}$ satisfies

$$u_n = \lfloor (n+1)\alpha + x_0 \rfloor - \lfloor n\alpha + x_0 \rfloor \qquad \text{for all } n \in \mathbb{Z}, \tag{7}$$

or

$$u_n = \lceil (n+1)\alpha + x_0 \rceil - \lceil n\alpha + x_0 \rceil \qquad \text{for all } n \in \mathbb{Z}, \tag{8}$$

where $\alpha \in (0,1)$ is an irrational number called the slope, and $x_0 \in [0,1)$ is called the intercept of u. In the former case, $(u_n)_{n\in\mathbb{Z}}$ is the so-called upper mechanical word, in the latter case the lower mechanical word, with slope α and intercept x_0.

If $(u_n)_{n\in\mathbb{Z}}$ is of the form (7) then, obviously,

$$u_n = \begin{cases} 0 & \text{if } \{n\alpha + x_0\} \in [0, 1-\alpha)\,, \\ 1 & \text{if } \{n\alpha + x_0\} \in [1-\alpha, 1)\,, \end{cases} \tag{9}$$

where $\{x\}$ denotes the fractional part of x, i.e., $\{x\} = x - \lfloor x \rfloor$. We can define a transformation $T : [0,1) \to [0,1)$ by the prescription

$$T(x) = \begin{cases} x + \alpha & \text{if } \{n\alpha + x_0\} \in [0, 1-\alpha) =: I_0\,, \\ x + \alpha - 1 & \text{if } \{n\alpha + x_0\} \in [1-\alpha, 1) =: I_1\,, \end{cases} \tag{10}$$

which satisfies $T(x) = \{x+\alpha\}$. It follows easily that the n-th iteration of T is given as

$$T^n(x) = \{x + n\alpha\} \qquad \text{for all } n \in \mathbb{Z}. \tag{11}$$

Putting (9) and (11) together, we see that a sturmian word $(u_n)_{n\in\mathbb{Z}}$ can be defined using the transformation T by

$$u_n = \begin{cases} 0 & \text{if } T^n(x_0) \in I_0, \\ 1 & \text{if } T^n(x_0) \in I_1. \end{cases}$$

Hence a sturmian word is given by iterations of the intercept x_0 under the mapping T, that is, by the orbit of x_0 under T.

The action of the mapping T from (10) is illustrated on Figure 2.

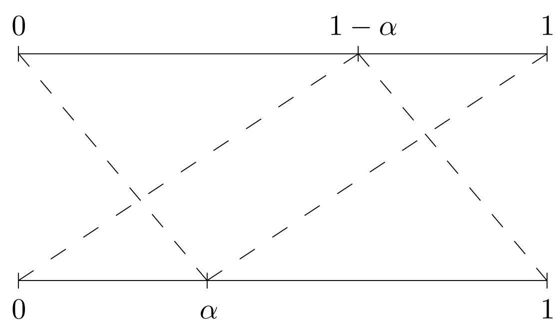

Figure 2: Graph of a 2-interval exchange transformation.

We see that T is in fact an exchange of two intervals $I_0 = [0, 1-\alpha)$ and $I_1 = [1-\alpha, 1)$. It is therefore called a 2-interval exchange transformation.

Let us mention that if $(u_n)_{n\in\mathbb{Z}}$ is an upper mechanical word, the corresponding 2-interval exchange transformation is given by $T : (0,1] \mapsto (0,1]$, with $I_0 = (0, 1-\alpha]$ and $I_1 = (1-\alpha, 1]$. Note also that it was not necessary that T was acting on a unit interval. We could choose an arbitrary interval divided into two parts, ratio of whose lengths would be irrational.

Analogically to the case of exchange of two intervals, we can define a 3-interval exchange transformation.

Definition. Let α, β, γ be three positive real numbers. Denote

$$\begin{array}{lcl} I_A & := & [0,\alpha) \\ I_B & := & [\alpha, \alpha+\beta) \\ I_C & := & [\alpha+\beta, \alpha+\beta+\gamma) \end{array} \qquad \text{or} \qquad \begin{array}{lcl} I_A & := & (0,\alpha] \\ I_B & := & (\alpha, \alpha+\beta] \\ I_C & := & (\alpha+\beta, \alpha+\beta+\gamma] \end{array}$$

respectively, and $I := I_A \cup I_B \cup I_C$. A mapping $T : I \to I$, given by

$$T(x) = \begin{cases} x + \beta + \gamma & \text{if } x \in I_A, \\ x - \alpha + \gamma & \text{if } x \in I_B, \\ x - \alpha - \beta & \text{if } x \in I_C, \end{cases} \tag{12}$$

is called a *3-interval exchange transformation* (3iet)[1] with parameters α, β, γ.

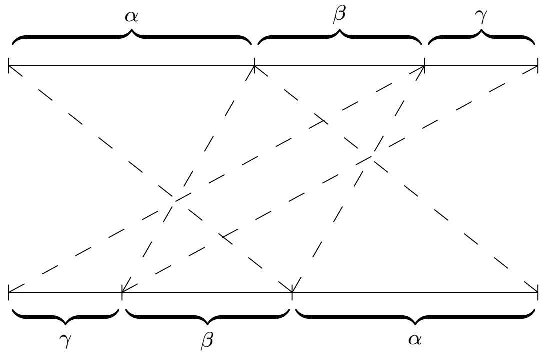

Figure 3: Graph of a 3-interval exchange transformation.

The graph of a 3-interval exchange is on Figure 3.

With a 3-interval exchange transformation T, one can naturally associate a ternary biinfinite word $u_T(x_0) = (u_n)_{n\in\mathbb{Z}}$, which codes the orbit of a point x_0 from the domain of T, as

$$u_n = \begin{cases} A & \text{if } T^n(x_0) \in I_A, \\ B & \text{if } T^n(x_0) \in I_B, \\ C & \text{if } T^n(x_0) \in I_C. \end{cases} \tag{13}$$

Similarly as in the case of a 2-interval exchange transformation, the infinite word coding a 3iet can be periodic or aperiodic, according to the choice of parameters α, β, γ. We will focus only on aperiodic words.

Definition. An aperiodic[2] word $u_T(x_0)$ coding the orbit of the point x_0 under the 3iet T defined above is called a *3iet word* with parameters α, β, γ and x_0.

The following lemma shows a close relation between words coding 3-interval exchange and 2-interval exchange transformations.

Lemma 2. *Let $u = (u_n)_{n\in\mathbb{Z}}$ be a word coding 3-interval exchange transformation and let $\sigma : \{A, B, C\}^* \to \{0, 1\}^*$ be a morphism given by*

$$A \mapsto 0\,, \qquad B \mapsto 01\,, \qquad C \mapsto 1\,. \tag{14}$$

Then $\sigma(u)$ codes a 2-interval exchange transformation.

Proof. Let u be the coding of x_0 under the 3-interval exchange transformation T with intervals $[0, \alpha)$, $[\alpha, \alpha + \beta)$ and $[\alpha + \beta, \alpha + \beta + \gamma)$.

Let S be the 2-interval exchange transformation of the intervals $I_0 = [0, \alpha + \beta)$ and $I_1 = [\alpha + \beta, \alpha + 2\beta + \gamma)$, i.e.,

$$S(x) = \begin{cases} x + \beta + \gamma & \text{if } x \in I_0, \\ x - \alpha - \beta & \text{if } x \in I_1. \end{cases}$$

[1] Note that the above defined mapping T should be more precisely called 3-interval exchange with the permutation (321), since the initial arrangement of intervals $I_A < I_B < I_C$ is changed to $T(C) < T(B) < T(A)$. Indeed, one can define also 3iet with a different permutation of intervals, e.g. (231). The corresponding 3iet word has the property that by changing all the letters C into B one obtains a sturmian word over the alphabet $\{A, B\}$. We will not consider such words.

[2] A biinfinite word $(u_n)_{n\in\mathbb{Z}}$ is called aperiodic if neither $u_0u_1u_2\cdots$ nor $\cdots u_{-3}u_{-2}u_{-1}$ is eventually periodic.

One can easily see that

$$
\begin{array}{lcl}
x \in [0,\alpha) & \Rightarrow & x \in I_0 \text{ and } T(x) = S(x)\,, \\
x \in [\alpha,\alpha+\beta) & \Rightarrow & x \in I_0,\ S(x) \in I_1 \text{ and } S^2(x) = T(x)\,, \\
x \in [\alpha+\beta,\alpha+\beta+\gamma) & \Rightarrow & x \in I_1, \text{ and } S(x) = T(x)\,.
\end{array}
$$

This proves that $\sigma(u)$ is the coding of x_0 under S. □

4 Periodic and aperiodic words coding 3iet

In order to clarify the relation between the parameters of a 3iet and the complexity of the corresponding infinite words, we recast the definition of these words in a new formalism. We show that every 3iet word codes distances in a discrete set arising as a projection of points of the lattice $\mathbb{Z}^2$. This construction is known as the cut-and-project method.

Let ε, η be real numbers, $\varepsilon \neq -\eta$. Every point $(a,b) \in \mathbb{Z}^2$ can be written in the form

$$(a,b) = (a+b\eta)\vec{x}_1 + (a-b\varepsilon)\vec{x}_2\,,$$

where

$$\vec{x}_1 = \frac{1}{\varepsilon+\eta}(\varepsilon,1) \qquad \text{and} \qquad \vec{x}_2 = \frac{1}{\varepsilon+\eta}(\eta,-1)\,.$$

Let V_1 and V_2 denote the lines in $\mathbb{R}^2$ spanned by $\vec{x}_1$ and $\vec{x}_2$, respectively. Then $(a+b\eta)\vec{x}_1$ is the projection of the lattice point (a,b) on V_1, whereas $(a-b\varepsilon)\vec{x}_1$ is its projection on V_2. Let Ω be a bounded interval. Then the set

$$\Sigma_{\varepsilon,\eta}(\Omega) := \{a+b\eta \mid a,b \in \mathbb{Z},\ a-b\varepsilon \in \Omega\} \tag{15}$$

is called the *Cut-and-project (C&P) set* with parameters $\varepsilon, \eta, \Omega$. Thus C&P sets arise by projection on the line V_1 of points of $\mathbb{Z}^2$ having their second projection in a chosen segment on V_2.

Proposition 3. *Let α, β, γ be positive real numbers, and let $T : [0,\alpha+\beta+\gamma) \mapsto [0,\alpha+\beta+\gamma)$ be a 3iet defined by* (12)*. Let $x_0 \in [0,\alpha+\beta+\gamma)$ and let $u_T(x_0) = (u_n)_{n\in\mathbb{Z}}$ be the biinfinite word given by* (13)*. Put*

$$\varepsilon := \frac{\beta+\gamma}{\alpha+2\beta+\gamma}\,, \quad l := \frac{\alpha+\beta+\gamma}{\alpha+2\beta+\gamma}\,, \quad c := \frac{x_0}{\alpha+2\beta+\gamma} \quad \textit{and} \quad \Omega = (c-l,c]\,, \tag{16}$$

and choose arbitrary $\eta > 0$. Then the C&P set $\Sigma_{\varepsilon,\eta}(\Omega)$ is a discrete set with the following properties:

1. *$0 \in \Sigma_{\varepsilon,\eta}(\Omega)$;*
2. *the distances between adjacent elements of $\Sigma_{\varepsilon,\eta}(\Omega)$ take values $\mu_A = \eta$, $\mu_B = 1+2\eta$, and $\mu_C = 1+\eta$;*
3. *the ordering of the distances with respect to the origin is coded by the word $u_T(x_0)$;*
4. *$\Sigma_{\varepsilon,\eta}(\Omega) = \big\{\lfloor c+n\varepsilon\rfloor + n\eta \mid n \in \mathbb{Z},\ \{c+n\varepsilon\} \in [0,l)\big\}$.*

Proof. The parameters ε, l, and c satisfy clearly

$$\varepsilon \in (0,1)\,, \quad \max\{\varepsilon, 1-\varepsilon\} < l \le 1\,, \quad 0 \in (c-l, c]\,. \tag{17}$$

The condition in (15) determining whether a given point $a + b\eta$ belongs to the C&P set $\Sigma_{\varepsilon,\eta}(\Omega)$ can be rewritten

$$a - b\varepsilon \in \Omega \quad \Leftrightarrow \quad c + b\varepsilon - l < a \le c + b\varepsilon \quad \Leftrightarrow \quad a = \lfloor c + b\varepsilon \rfloor \text{ and } \{c + b\varepsilon\} \in [0, l)\,.$$

Therefore, the C&P set $\Sigma_{\varepsilon,\eta}(\Omega)$ can be expressed as

$$\Sigma_{\varepsilon,\eta}(\Omega) = \big\{ \lfloor c + n\varepsilon \rfloor + n\eta \mid n \in \mathbb{Z},\ \{c + n\varepsilon\} \in [0, l) \big\}\,. \tag{18}$$

Let us denote $y_n := \lfloor c + n\varepsilon \rfloor + n\eta$ and $y_n^* := \{c + n\varepsilon\}$. From the choice of the parameter ε and η we can derive that the sequence $(y_n)_{n\in\mathbb{Z}}$ is strictly increasing. Since $\Sigma_{\varepsilon,\eta}(\Omega) \subset \{y_n \mid n \in \mathbb{Z}\}$, to every element $y \in \Sigma_{\varepsilon,\eta}(\Omega)$ corresponds a point y^*. We show that the distance of y and its right neighbour depends on the position of y^* in the interval $[0, l)$. Moreover, if z is the right neighbour of y in $\Sigma_{\varepsilon,\eta}(\Omega)$, then $z^* = \widetilde{T}(y^*)$, where $\widetilde{T} : [0, l) \to [0, l)$ is a 3iet given by the prescription

$$\widetilde{T}(x) = \begin{cases} x + \varepsilon & \text{if } x \in [0, l - \varepsilon)\,, \\ x + 2\varepsilon - 1 & \text{if } x \in [l - \varepsilon, 1 - \varepsilon)\,, \\ x + \varepsilon - 1 & \text{if } x \in [1 - \varepsilon, l)\,. \end{cases} \tag{19}$$

Let us determine the right neighbour of a point $y \in \Sigma_{\varepsilon,\eta}(c - l, c]$. Let $y = y_n$, $n \in \mathbb{Z}$, i.e., $y_n^* = \{c + n\varepsilon\} \in [0, l)$. We discuss three separate cases, all the time using the fact that $\max\{\varepsilon, 1 - \varepsilon\} < l \le 1$.

i) if $y_n^* \in [0, l - \varepsilon)$ then $y_{n+1}^* = \{c + (n+1)\varepsilon\} = y_n^* + \varepsilon \in [0, l)$ and $\lfloor c + n\varepsilon \rfloor = \lfloor c + (n+1)\varepsilon \rfloor$. Hence the distance between y_n and its right neighbour is $y_{n+1} - y_n = \eta$.

ii) if $y_n^* \in [l - \varepsilon, 1 - \varepsilon)$ then $y_{n+1}^* = \{c + (n+1)\varepsilon\} = y_n^* + \varepsilon \in [l, 1)$, hence y_{n+1} does not belong to the set $\Sigma_{\varepsilon,\eta}(c - l, c]$. However, $y_{n+2}^* = \{c + (n+2)\varepsilon\} = y_n^* + 2\varepsilon - 1 \in [0, l)$ and $\lfloor c + (n+2)\varepsilon \rfloor = 1 + \lfloor c + n\varepsilon \rfloor$. Therefore the right neighbour of y_n is y_{n+2} and we have $y_{n+2} - y_n = 1 + 2\eta$.

iii) if $y_n^* \in [1 - \varepsilon, l)$ then $y_{n+1}^* = \{c + (n+1)\varepsilon\} = y_n^* + \varepsilon - 1 \in [0, l)$, y_{n+1} is the right neighbour of y_n and $y_{n+1} - y_n = 1 + \eta$.

As $y_0 = 0 \in \Sigma_{\varepsilon,\eta}(c - l, c]$ and $y_0^* = \{c\} = c$, the distances between consecutive elements of the C&P set $\Sigma_{\varepsilon,\eta}(c - l, c]$ are coded by the infinite word $u_{\widetilde{T}}(c)$. It is easy to see that with our choice of l, ε, and c, the lengths of the partial intervals in the definition of the 3iet $\widetilde{T}$ and the starting point c are only $(\alpha + 2\beta + \gamma)$-multiples of the partial intervals of the 3iet T and its starting point x_0, ($\widetilde{T}$ and T are homothetic 3iets). Therefore $u_{\widetilde{T}}(c) = u_T(x_0)$. ☐

Let us mention that a 3iet T with the domain $(0, \alpha + \beta + \gamma]$ corresponds also to a C&P set with parameters similar to (16).

It is known that a word coding an r-interval exchange transformation with arbitrary permutation of intervals has complexity $\mathcal{C}(n) \le (r - 1)n + 1$ for all $n \in \mathbb{N}$, see [15]. It is useful to distinguish the words with full complexity and the others.

Definition. A 3iet word is called *non-degenerated*, if $\mathcal{C}(n) = 2n + 1$ for all $n \in \mathbb{N}$. Otherwise it is called *degenerated*.

The following proposition allows one to classify the words coding 3iet according to the parameters to periodic, 3iet degenerate, and 3iet non-degenerate infinite words.

Proposition 4. *Let T be a 3iet transformation of the interval I with parameters α, β, γ, and let $x_0 \in I$.*

- *The infinite word $u_T(x_0)$ defined by (13) is aperiodic if and only if*

$$\alpha + \beta \text{ and } \beta + \gamma \text{ are linearly independent over } \mathbb{Q}.$$

- *If the word $u_T(x_0)$ is aperiodic then it is degenerated if and only if*

$$\alpha + \beta + \gamma \in (\alpha + \beta)\mathbb{Z} + (\beta + \gamma)\mathbb{Z}\,.$$

Proof. The formula (18) for the C&P set $\Sigma_{\varepsilon,\eta}(c - l, c]$ implies easily that if ε is rational, then the set $\Sigma_{\varepsilon,\eta}(\Omega)$ is periodic, i.e., the orbit of every point under the 3iet $\widetilde{T}$ is periodic. On the other hand, if ε is irrational, the sequence $\{c + n\varepsilon\}$ is uniformly distributed, and thus also the orbit of every point under $\widetilde{T}$ is dense in $[0, l)$. The relation (16) between the parameters ε and α, β, γ implies the statement about periodicity of $u_T(x_0)$.

The complexity of an infinite word coding a C&P set with irrational parameters ε, η has been described in [11]. It is shown that such a word has the complexity $\mathcal{C}(n) = 2n + 1$ for all n if and only if the length l of the interval Ω from (15) satisfies $l \notin \mathbb{Z} + \mathbb{Z}\varepsilon$. The relation (16) implies the necessary and sufficient condition for the degeneracy of the corresponding infinite word. □

We will use the following reformulation of the above statements.

Corollary 5. *The infinite word $u_T(x_0)$, defined by (13), with parameters $\alpha, \beta, \gamma > 0$ is*

- *periodic if there exist $K, L \in \mathbb{Z}$, $K, L \neq 0$ such that*

$$(\alpha, \beta, \gamma) \begin{pmatrix} K \\ K+L \\ L \end{pmatrix} = 0\,, \tag{20}$$

- *aperiodic degenerate if there exist unique $K, L \in \mathbb{Z}$ such that*

$$(\alpha, \beta, \gamma) \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = (\alpha, \beta, \gamma) \begin{pmatrix} K \\ K+L \\ L \end{pmatrix}\,. \tag{21}$$

Note that the sequence $\{c + n\varepsilon\}$ being uniformly distributed for ε irrational implies not only the aperiodicity of the infinite word, but also that the densities of letters are well defined.

Corollary 6. *All letters in a 3iet word u with parameters α, β, γ have a well defined density and the vector of densities of u, denoted by $\vec{\rho}_u := \big(\rho(A), \rho(B), \rho(C)\big)$, is proportional to the vector (α, β, γ).*

For the transformation T of exchange of r intervals, it is generally difficult to describe the conditions under which the corresponding dynamical system is minimal, i.e., under which condition the orbit $\{T^n(x_0) \mid n \in \mathbb{Z}\}$ of any point x_0 is dense in the domain of T. Keane provides in [15] two sufficient conditions for the minimality of T: one of them is the linear independence of parameters α, β and γ over $\mathbb{Q}$; second, weaker condition is that the orbits of all discontinuity points of T are disjoint. This condition is called i.d.o.c. In [10] it is shown that the parameters α, β, γ fulfill i.d.o.c. if and only if they satisfy neither (20) nor (21). Nevertheless, even the weaker condition i.d.o.c. is only sufficient, but not necessary for the minimality of the dynamical system of T. The geometric representation of 3iet T using a cut-and-project set allows us to provide a simple characterization of minimal dynamical systems among 3iet.

Corollary 7. *The dynamical system given by a 3-interval exchange transformation T with parameters α, β, γ is minimal if and only if the numbers $\alpha+\beta$ and $\beta+\gamma$ are linearly independent over $\mathbb{Q}$.*

***Remark* 8.** It can be shown, (see [2, 10, 11]), that a 3iet word is degenerated if and only if the orbits of the two discontinuity points of the corresponding 3iet T have a non-empty intersection, formally, $\{T^n(\alpha) \mid n \in \mathbb{Z}\} \cap \{T^n(\alpha+\beta) \mid n \in \mathbb{Z}\} \neq \emptyset$. The complexity of a degenerate 3iet word is $\mathcal{C}(n) = n + \mathit{const}$ for sufficiently large n. Cassaigne [8] calls one-sided infinite words with such complexity quasi-sturmian words. By a slight modification of his results one can show that for any 3iet word u with complexity $\mathcal{C}_u(n) \leq n + \text{const}$ there exists a sturmian word $(v_n)_{n\in\mathbb{Z}}$ over $\{0, 1\}$ and finite words $w_1, w_2 \in \{A, B, C\}^*$ such that

$$u = \cdots w_{v_{-2}} w_{v_{-1}} \mid w_{v_0} w_{v_1} w_{v_2} \cdots ,$$

that is, u is obtained from v by applying the morphism $0 \mapsto w_0$ and $1 \mapsto w_1$.

5 Morphisms preserving 3iet words

Definition. A morphism on the alphabet $\{A, B, C\}$ is said to be *3iet preserving* if $\varphi(u)$ is a 3iet word for every 3iet word u.

Let us recall that 3iet words are defined as those words coding 3-interval exchange transformations, which are aperiodic. Similarly, sturmian words are aperiodic words coding 2-interval exchange transformations.

In the rest of this section we give several useful examples of 3iet preserving morphisms.

***Example* 9.** We will prove that the morphism φ over $\{A, B, C\}$ given by prescriptions

$$A \mapsto AC\,, \qquad B \mapsto BC\,, \qquad C \mapsto C\,, \tag{22}$$

is 3iet preserving. Let us consider an arbitrary 3iet word u with arbitrary parameters α, β, γ and x_0. The corresponding transformation T is given by (12). We show that the infinite word $\varphi(u)$ is a 3iet word, namely the one with parameters $\alpha' = \alpha, \beta' = \beta, \gamma' = \alpha + \beta + \gamma$ and $x'_0 = x_0$.

The transformation T' (see Figure 4) corresponding to the parameters α', β', γ' is given by

$$T'(x) = \begin{cases} x + \alpha + 2\beta + \gamma & \text{if } x \in [0, \alpha) =: I'_A, \\ x + \beta + \gamma & \text{if } x \in [\alpha, \alpha + \beta) =: I'_B, \\ x - \alpha - \beta & \text{if } x \in [\alpha + \beta, 2\alpha + 2\beta + \gamma) =: I'_C. \end{cases} \tag{23}$$

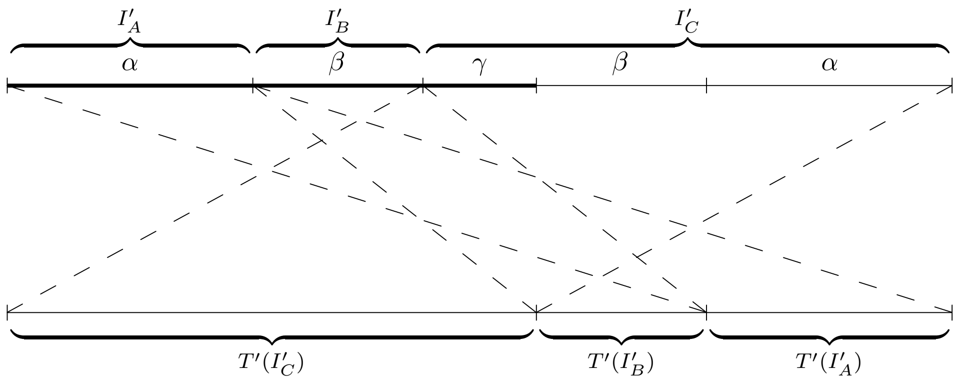

Figure 4: Graph of the transformation T'.

Obviously, (12) and (23) imply for a point $x \in I_A = I'_A$ that

$$T'(x) = x + \alpha + 2\beta + \gamma \in I'_C\,,$$
$$(T')^2(x) = x + \beta + \gamma = T(x)\,.$$

Hence any point $x \in I_A$ belongs in the new 3iet to the interval I'_A, its first iteration is $T'(x) \in I'_C$ and the second iteration $(T')^2(x)$ sends to the same place as the first iteration of the original transformation T. Therefore we substitute $A \mapsto AC$. Similarly, for a point $x \in I_B = I'_B$ we have

$$T'(x) = x + \beta + \gamma \in I'_C\,,$$
$$(T')^2(x) = x - \alpha + \gamma = T(x)\,,$$

and so $B \mapsto BC$. Finally, for $x \in I_C \subsetneqq I'_C$ we get $T'(x) = T(x)$ and hence $C \mapsto C$. Thus we see that the 3iet word coding x'_0 under T' coincides with the word $\varphi(u)$.

***Example* 10.** It is easy to see that the morphism ξ over $\{A, B, C\}$ given by prescriptions

$$A \mapsto C\,, \qquad B \mapsto B\,, \qquad C \mapsto A\,, \tag{24}$$

is a 3iet preserving morphism. To a 3iet word, which codes the orbit of x_0 under the transformation T with intervals $[0, \alpha) \cup [\alpha, \alpha+\beta) \cup [\alpha+\beta, \alpha+\beta+\gamma)$, is assigns a 3iet word, which codes the orbit of $\alpha+\beta+\gamma-x_0$ under the transformation $\tilde{T}$ with intervals $(0, \gamma] \cup (\gamma, \gamma+\beta] \cup (\gamma+\beta, \gamma+\beta+\alpha]$.

***Example* 11.** Let us consider the morphism φ_0 on $\{A, B, C\}$ given by $A \mapsto B$, $B \mapsto BCB$ and $C \mapsto CAC$. It is a primitive morphism with $\det \boldsymbol{M}_{\varphi_0} = 1$ and $\boldsymbol{M}^3_{\varphi_0} > 0$. Let u be an arbitrary 3iet word with parameters α, β, γ. Using the same technique as in Example 9 one can show that φ_0 is 3iet preserving; the 3iet word coinciding with $\varphi_0(u)$ has parameters $\alpha' = \gamma$, $\beta' = \beta+\alpha+\beta$, $\gamma' = \gamma + \beta + \gamma$.

6 Proof of Theorem A

The aim of this section is to prove that the matrix $\boldsymbol{M}$ of a 3iet preserving morphism fulfills the following condition

$$\boldsymbol{M}\boldsymbol{E}\boldsymbol{M}^T = \pm\boldsymbol{E}, \qquad \text{where } \boldsymbol{E} = \begin{pmatrix} 0 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & -1 & 0 \end{pmatrix}. \tag{25}$$

The main tool used in the proof of this property of $\boldsymbol{M}$ is the fact that the matrix of a sturmian morphism has determinant ± 1 and some auxiliary statements formulated as Lemma 12 and Lemma 14.

Lemma 12. *Let φ be a 3iet preserving morphism and $\boldsymbol{M}$ its incidence matrix. Let $\mathcal{P}$ be a subspace of $\mathbb{R}^3$ spanned by the vectors $\left(\begin{smallmatrix}1\\1\\0\end{smallmatrix}\right)$, $\left(\begin{smallmatrix}0\\1\\1\end{smallmatrix}\right)$. Then $\boldsymbol{M}\mathcal{P} = \mathcal{P}$.*

Proof. If u is a 3iet word with parameters (α, β, γ), then according to (5) and Corollary 6, $\varphi(u)$ is a 3iet word with parameters $(\alpha, \beta, \gamma)\boldsymbol{M}$. Since φ is a 3iet preserving morphism, it means that $\varphi(u)$ is aperiodic, whenever u is aperiodic. With the help of Corollary 5, it implies that for every pair $K, L \in \mathbb{Z} \setminus \{0\}$ and every triple of positive numbers (α, β, γ), we have

$$(\alpha,\beta,\gamma)\boldsymbol{M}\left(\begin{smallmatrix}K\\K+L\\L\end{smallmatrix}\right) = 0 \quad \Longrightarrow \quad \exists\, H, S \in \mathbb{Z}\setminus\{0\} \text{ such that } (\alpha,\beta,\gamma)\left(\begin{smallmatrix}H\\H+S\\S\end{smallmatrix}\right) = 0\,. \tag{26}$$

Since $\left\{\boldsymbol{M}\left(\begin{smallmatrix}K\\K+L\\L\end{smallmatrix}\right) \;\middle|\; K, L \in \mathbb{Z}\right\}$ is a 2-dimensional lattice in $\mathbb{R}^3$, there exist two linearly independent pairs K_1, L_1, K_2, L_2 such that $\boldsymbol{M}\left(\begin{smallmatrix}K_i\\K_i+L_i\\L_i\end{smallmatrix}\right)$, $i = 1, 2$, have both positive and negative components, and therefore for both $i = 1, 2$, there exist infinitely many triples (α, β, γ) such that $(\alpha,\beta,\gamma)\boldsymbol{M}\left(\begin{smallmatrix}K_i\\K_i+L_i\\L_i\end{smallmatrix}\right) = 0$. This, together with (26), implies

$$\boldsymbol{M}\left(\begin{smallmatrix}K_i\\K_i+L_i\\L_i\end{smallmatrix}\right) = const.\left(\begin{smallmatrix}H_i\\H_i+S_i\\S_i\end{smallmatrix}\right)\,, \qquad \text{for some } H_i, S_i \in \mathbb{Z}\setminus\{0\},\ i = 1, 2. \tag{27}$$

Consequently, $\boldsymbol{M}\mathcal{P} \subseteq \mathcal{P}$. We now show that $\boldsymbol{M}\mathcal{P} = \mathcal{P}$. Suppose the opposite, i.e., that $\boldsymbol{M}\left(\begin{smallmatrix}1\\1\\0\end{smallmatrix}\right)$ and $\boldsymbol{M}\left(\begin{smallmatrix}0\\1\\1\end{smallmatrix}\right)$ are linearly dependent. Then there exist $K, L \in \mathbb{Z} \setminus \{0\}$ such that

$$\left(\begin{smallmatrix}0\\0\\0\end{smallmatrix}\right) = K\boldsymbol{M}\left(\begin{smallmatrix}1\\1\\0\end{smallmatrix}\right) + L\boldsymbol{M}\left(\begin{smallmatrix}0\\1\\1\end{smallmatrix}\right) = \boldsymbol{M}\left(\begin{smallmatrix}K\\K+L\\L\end{smallmatrix}\right).$$

This, however, implies that for arbitrary parameters (α, β, γ), we have

$$(\alpha,\beta,\gamma)\boldsymbol{M}\left(\begin{smallmatrix}K\\K+L\\L\end{smallmatrix}\right) = 0\,,$$

i.e., the word $\varphi(u)$ is periodic for arbitrary 3iet word u, which is a contradiction with the assumption that φ is a 3iet preserving morphism. □

***Remark* 13.** Denote

$$\vec{x}_1 := \left(\begin{smallmatrix}1\\1\\0\end{smallmatrix}\right), \qquad \vec{x}_2 := \left(\begin{smallmatrix}0\\1\\1\end{smallmatrix}\right), \qquad \vec{x}_3 := \left(\begin{smallmatrix}0\\1\\0\end{smallmatrix}\right).$$

The triplet of vectors $\vec{x}_1$, $\vec{x}_2$, $\vec{x}_3$ forms a basis of $\mathbb{R}^3$. Denoting $\boldsymbol{P} = \left(\begin{smallmatrix}1&0&0\\1&1&1\\0&1&0\end{smallmatrix}\right)$, we have $\det \boldsymbol{P} = 1$, and thus $\vec{x}_1$, $\vec{x}_2$, $\vec{x}_3$ is also a basis of the integer lattice $\mathbb{Z}^3$. In the same time, the pair $\vec{x}_1$, $\vec{x}_2$ is a basis of the invariant subspace $\mathcal{P}$ of the matrix $\boldsymbol{M}$. We have

$$\boldsymbol{P}^{-1} = \left(\begin{smallmatrix}1&0&0\\0&0&1\\-1&1&-1\end{smallmatrix}\right) \quad \text{and} \quad \boldsymbol{P}^{-1}\boldsymbol{M}\boldsymbol{P} = \begin{pmatrix} m_{11}+m_{12} & m_{12}+m_{13} & m_{12}\\ m_{31}+m_{32} & m_{32}+m_{33} & m_{32}\\ 0 & 0 & -m_{12}+m_{22}-m_{32}\end{pmatrix},$$

where the 0's in the third row correspond to the fact that $\boldsymbol{P}^{-1}\boldsymbol{M}\boldsymbol{P}$ can be seen as the matrix $\boldsymbol{M}$ written in the basis $\vec{x}_1$, $\vec{x}_2$, $\vec{x}_3$, where the first two vectors form a basis of the invariant subspace $\mathcal{P}$. Since $\boldsymbol{M}\mathcal{P} = \mathcal{P}$, we have

$$\det \begin{pmatrix} m_{11} + m_{12} & m_{12} + m_{13} \\ m_{31} + m_{32} & m_{32} + m_{33} \end{pmatrix} \neq 0\,.$$

Lemma 14. *Let $\boldsymbol{M} = (m_{ij})$ be the incidence matrix of a 3iet preserving morphism φ. Then*

$$\det \begin{pmatrix} m_{11} + m_{12} & m_{12} + m_{13} \\ m_{31} + m_{32} & m_{32} + m_{33} \end{pmatrix} = \delta \in \{1, -1\}\,. \tag{28}$$

Proof. Let us choose a sturmian word $u \in \{A, C\}^{\mathbb{Z}}$ and a sequence $(u^{(m)})_{m\in\mathbb{N}}$ of 3iet words such that $u = \lim_{m\to\infty} u^{(m)}$. For example, let u be the coding of $x_0 = 0$ under the 2-interval exchange transformation T with $I_0 = [0, 1-\alpha)$ and $I_1 = [1-\alpha, 1)$, where α is an arbitrary irrational number. Then we can choose $u^{(m)}$ to be the 3iet word that codes $x_0 = 0$ under the 3-interval exchange transformation with intervals $I_A = [0, 1-\alpha-\frac{1}{m})$, $I_B = [1-\alpha-\frac{1}{m}, 1-\alpha)$ and $I_C = [1-\alpha, 1)$.

Let σ be a morphism given by

$$A \mapsto A\,, \quad B \mapsto AC\,, \quad C \mapsto C\,.$$

Since any morphism on $\{A, B, C\}^{\mathbb{Z}}$ is a continuous mapping, we have

$$(\sigma \circ \varphi)(u^{(m)}) \to (\sigma \circ \varphi)(u)\,.$$

According to the assumption, the morphism φ is 3iet preserving, hence $\varphi(u^{(m)})$ are 3iet words. By Lemma 2, the words $(\sigma \circ \varphi)(u^{(m)})$, $m \in \mathbb{N}$, code 2-interval exchange transformations, and by Lemma 1, the limit of these words, that is the word $(\sigma \circ \varphi)(u)$, is either sturmian or the densities of its letters are rational.

The matrix of σ is $\left(\begin{smallmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{smallmatrix}\right)$, which implies by (4) that the matrix of $\sigma \circ \varphi$ is

$$\boldsymbol{M}_{\sigma\circ\varphi} = \begin{pmatrix} m_{11} + m_{12} & 0 & m_{12} + m_{13} \\ m_{21} + m_{22} & 0 & m_{22} + m_{23} \\ m_{31} + m_{32} & 0 & m_{32} + m_{33} \end{pmatrix}.$$

Since $\sigma \circ \varphi$ maps a sturmian word u over $\{A, C\}$ to a word over the same alphabet, we are interested only in the matrix of this morphism over $\{A, C\}$, that is,

$$\widetilde{\boldsymbol{M}} = \begin{pmatrix} m_{11} + m_{12} & m_{12} + m_{13} \\ m_{31} + m_{32} & m_{32} + m_{33} \end{pmatrix}. \tag{29}$$

Let us suppose that the densities of A and C in u are $1-\alpha$ and α, respectively. Using (5) we find the density of A in $(\sigma \circ \varphi)(u)$ to be

$$\rho(A) = \frac{(1-\alpha, \alpha)\ \widetilde{\boldsymbol{M}} \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)}{(1-\alpha, \alpha)\ \widetilde{\boldsymbol{M}} \left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right)}\,. \tag{30}$$

If $\rho(A)$ is irrational, the word $(\sigma\circ\varphi)(u)$ is sturmian and hence the morphism $\sigma\circ\varphi$ is sturmian. This implies $\det \widetilde{\boldsymbol{M}} = \pm 1$.

The irrational number α, i.e., the density of A in the sturmian word u, was chosen arbitrarily. Therefore $\rho(A)$, given by (30), will be rational for any irrational α only in case when

$$p\widetilde{\boldsymbol{M}}\begin{pmatrix}1\\0\end{pmatrix} = q\widetilde{\boldsymbol{M}}\begin{pmatrix}1\\1\end{pmatrix}, \quad \text{for some } \ p,q\in\mathbb{Z}\setminus\{0\}\,. \tag{31}$$

This however implies that the matrix $\widetilde{\boldsymbol{M}}$ is singular, which contradicts Remark 13. □

We are now in position to finish the proof of Theorem A.

Theorem A. *Let $\boldsymbol{M}$ be the incidence matrix of a 3iet preserving morphism. Then*

$$\boldsymbol{M}\boldsymbol{E}\boldsymbol{M}^T = \pm\boldsymbol{E}, \quad \textit{where} \ \ \boldsymbol{E} = \begin{pmatrix}0&1&1\\-1&0&1\\-1&-1&0\end{pmatrix}. \tag{32}$$

Proof. Using the notation of Remark 13 for the matrix $\boldsymbol{P}$, we obviously see that the matrix $\boldsymbol{P}^{-1}\boldsymbol{M}\boldsymbol{P}$ has $(0,0,-1)$ for its left eigenvector corresponding to the eigenvalue $-m_{12}+m_{22}-m_{32}$. It is then trivial to verify that $(0,0,-1)\boldsymbol{P}^{-1} = (1,-1,1)$ is a left eigenvector of the matrix $\boldsymbol{M}$ corresponding to the same eigenvalue. Since

$$\det\boldsymbol{M} = \det(\boldsymbol{P}^{-1}\boldsymbol{M}\boldsymbol{P}) = \delta(-m_{12}+m_{22}-m_{32})\,, \tag{33}$$

where $\delta\in\{-1,1\}$ is given by (28), we derive that $(1,-1,1)$ is a left eigenvector of the matrix $\boldsymbol{M}$ corresponding to the eigenvalue $\delta\det\boldsymbol{M}$. Denoting $\Delta := \det\boldsymbol{M}$, we can write

$$(1,-1,1)\boldsymbol{M} = \delta\Delta(1,-1,1)\,. \tag{34}$$

This implies that the matrix $\boldsymbol{M}$ can be written in the following form,

$$\boldsymbol{M} = \begin{pmatrix} m_{11} & m_{12} & m_{13}\\ m_{11}+m_{31}-\delta\Delta & m_{12}+m_{32}+\delta\Delta & m_{13}+m_{33}-\delta\Delta\\ m_{31} & m_{32} & m_{33}\end{pmatrix}. \tag{35}$$

With this, one can verify by inspection, that $\boldsymbol{M}\boldsymbol{E}\boldsymbol{M}^T = \delta\boldsymbol{E}$, using Lemma 14 for simplification of algebraic expressions. □

As a partial result, we have shown in the above proof the following interesting statement.

Corollary 15. *Let $\boldsymbol{M}$ be the matrix of a 3iet preserving morphism φ. Then the vector $(1,-1,1)$ is a left eigenvector of $\boldsymbol{M}$, associated with the eigenvalue $\det\boldsymbol{M}$ or $-\det\boldsymbol{M}$, i.e.,*

$$(1,-1,1)\boldsymbol{M} = \pm\det\boldsymbol{M}(1,-1,1)\,. \tag{36}$$

The other eigenvalues λ_1 and λ_2 of the matrix $\boldsymbol{M}$ are either quadratic mutually conjugate algebraic units, or $\lambda_1,\lambda_2\in\{1,-1\}$.

From the form (35) of the matrix $\boldsymbol{M}$ we derive the following Corollary.

Corollary 16. *Let $\boldsymbol{M}$ be a matrix of a 3iet preserving morphism. Then the sum of its first and the third row differs from the sum of its second row by $\pm\det\boldsymbol{M}$. Formally,*

$$(1,0,1)\boldsymbol{M}\begin{pmatrix}1\\1\\1\end{pmatrix} - (0,1,0)\boldsymbol{M}\begin{pmatrix}1\\1\\1\end{pmatrix} = \pm\det\boldsymbol{M}\,.$$

7 3iet preserving morphisms versus fixed points

The proof of Theorem B, which is performed in Section 8 is based on the properties of 3iet words, which are fixed points of morphisms. In this section we therefore inspect, which 3iet preserving morphisms have a fixed point.

A fixed point of a morphism φ over an alphabet $\mathcal{A}$ is the limit $\lim_{n\to\infty}\varphi^n(a_i)|\varphi^n(a_j)$ for some $a_i, a_j \in \mathcal{A}$. Similarly to the case of sturmian words, the set of 3iet words is not compact, and therefore in general the accumulation point u of a sequence $(u^{(m)})_{m\in\mathbb{N}}$ of 3iet words is not necessarily a 3iet word. The special case when the accumulation point belongs to the set of 3iet words is treated by the following Lemma.

Lemma 17. *Let α, β, γ be positive real numbers such that $\alpha+\beta$ and $\beta+\gamma$ are linearly independent over $\mathbb{Q}$. Let T_1, T_2 be the 3iet transformations with parameters α, β, γ and domain $[0, \alpha+\beta+\gamma)$, $(0, \alpha+\beta+\gamma]$, respectively. Let $(u^{(n)})_{n\in\mathbb{N}}$ be a sequence of 3iet words and $(x^{(n)})_{n\in\mathbb{N}}$ a sequence of points in $[0, \alpha+\beta+\gamma]$ such that*

- *$u^{(n)} = u_{T_1}(x^{(n)})$ or $u^{(n)} = u_{T_2}(x^{(n)})$ for all $n \in \mathbb{N}$;*
- *$x^{(n)}$ is a monotonous sequence with the limit x.*

Then $\lim_{n\to\infty} u^{(n)}$ exists and is equal to the 3iet word $u_{T_1}(x)$ or $u_{T_2}(x)$.

Proof. We use a statement from [11]. For a given m put

$$D_m := \{T_1^i(\alpha), T_1^i(\alpha+\beta), T_2^i(\alpha), T_2^i(\alpha+\beta) \mid -m \le i \le m\}\,.$$

Let $a < b$ and let $(a, b) \cap D_m = \emptyset$. Then for all $z \in (a, b)$ we have

$$\mathrm{d}\big(u_{T_1}(a), u_{T_1}(z)\big) < \frac{1}{1+m}\,, \qquad \mathrm{d}\big(u_{T_1}(a), u_{T_2}(z)\big) < \frac{1}{1+m}\,, \tag{37}$$

$$\mathrm{d}\big(u_{T_2}(b), u_{T_1}(z)\big) < \frac{1}{1+m}\,, \qquad \mathrm{d}\big(u_{T_2}(b), u_{T_2}(z)\big) < \frac{1}{1+m}\,. \tag{38}$$

Assume that the sequence $(x^{(n)})_{n\in\mathbb{N}}$ is decreasing. For $\varepsilon > 0$, we find $m \in \mathbb{N}$ such that $\varepsilon > \frac{1}{m+1}$ and we put $\delta_m =: \sup\{y > x \mid y \notin D_m\}$. Since $x^{(n)} \searrow x$, there exists n_0 such that for all $n > n_0$ we have $x \le x^{(n)} < x + \delta_m$. Since $u^{(n)} = u_{T_1}(x^{(n)})$ or $u^{(n)} = u_{T_2}(x^{(n)})$, we obtain, using (37) for the interval $(a, b) = (x, x+\delta_m)$, that $\mathrm{d}\big(u_{T_1}(x), u^{(n)}\big) < \varepsilon$, which implies $\lim_{n\to\infty} u^{(n)} = u_{T_1}(x)$. Similarly we use (38) in the case $x^{(n)} \nearrow x$. □

***Remark* 18.** The assumption of primitivity of the morphism φ is essential in the above statement. For example, the morphism φ defined by (22) is 3iet preserving, yet the only fixed points of an arbitrary power φ^p, $p \in \mathbb{N}$, $p \ge 1$, are

$$\cdots CCC|ACCC\cdots\,, \qquad \cdots CCC|BCCC\cdots\,, \qquad \cdots CCC|CCC\cdots$$

The following proposition deals with the original aim of this section, namely with the search for 3iet preserving morphisms having 3iet words as their fixed points.

Proposition 19. *Let φ be a primitive 3iet preserving morphism. Then there exists $p \in \mathbb{N}$, $p \ge 1$, such that φ^p has a fixed point, and this fixed point is a 3iet word.*

Proof. Without loss of generality, we may assume that the incidence matrix $\boldsymbol{M}$ of the morphism φ is positive. Otherwise, we show the validity of the statement for $\psi = \varphi^k$ for some k, which implies the validity of the statement for φ.

Let (α, β, γ) be a positive left eigenvector of $\boldsymbol{M}$. First we show that an infinite word coding a 3iet with such parameters is not periodic. For contradiction, assume that (α, β, γ) satisfy (5), that is,

$$(\alpha, \beta, \gamma) \begin{pmatrix} K \\ K+L \\ L \end{pmatrix} = 0 , \quad \text{for some } K, L \in \mathbb{Z} \setminus \{0\}. \tag{39}$$

If the Perron eigenvalue λ_1 of $\boldsymbol{M}$ is a quadratic irrational number, one can assume without loss of generality that the components of the vector (α, β, γ) belong to the quadratic field $\mathbb{Q}(\lambda_1)$. For any $x \in \mathbb{Q}(\lambda_1)$, denote by x' the image of x under the Galois automorphism of $\mathbb{Q}(\lambda_1)$. Since the matrix $\boldsymbol{M}$ and the vector $\begin{pmatrix} K \\ K+L \\ L \end{pmatrix}$ have integer components, the vector $(\alpha', \beta', \gamma')$ is an eigenvector to the eigenvalue $\lambda_1' = \lambda_2$ and satisfies

$$(\alpha', \beta', \gamma') \begin{pmatrix} K \\ K+L \\ L \end{pmatrix} = 0 .$$

Using Corollary 15, the vector $(1, -1, 1)$ is a left eigenvector of $\boldsymbol{M}$ corresponding to the eigenvalue $\pm \det \boldsymbol{M}$. Therefore vectors $(\alpha', \beta', \gamma')$, (α, β, γ) and $(1, -1, 1)$ are eigenvectors of $\boldsymbol{M}$ corresponding to different eigenvalues, which means that they are linearly independent. All of them are orthogonal to the vector $\begin{pmatrix} K \\ K+L \\ L \end{pmatrix}$, which implies $K = L = 0$. This contradicts (39).

By Corollary 15, it remains to discuss the case when the Perron eigenvalue of $\boldsymbol{M}$ is $\lambda_1 = 1$. This is impossible due to the fact that a positive integral matrix $\boldsymbol{M}$ cannot have 1 as its eigenvalue corresponding to a positive eigenvector. Thus we have shown that the infinite word coding a 3iet with parameters α, β, γ is not periodic.

Denote T_1, T_2 the 3iet transformations with parameters α, β, γ and domain $[0, \alpha + \beta + \gamma)$, $(0, \alpha + \beta + \gamma]$, respectively.

Let $u^{(0)}$ be an arbitrary 3iet word coding the orbit of a point by T_1. Put

$$u^{(n)} := \varphi^n(u^{(0)}) , \quad \text{for } n \geq 1.$$

Since the vector of densities of $u^{(0)}$ is a left eigenvector of the incidence matrix of the morphism φ, every word $u^{(n)}$, $n \in \mathbb{N}$, has the same density of letters. As φ is a 3iet preserving morphism, the word $u^{(n)}$ is a 3iet word coding the orbit of a point under T_1 or T_2, for every $n \in \mathbb{N}$.

The space of infinite words over the alphabet $\{A, B, C\}$ is compact, and thus there exists a Cauchy subsequence of the sequence $(u^{(n)})_{n \in \mathbb{N}}$. Therefore there exist $m_0, n_0 \in \mathbb{N}$, $n_0 > m_0$, such that

$$\mathrm{d}\big(u^{(n_0)}, u^{(m_0)}\big) < \frac{1}{2} . \tag{40}$$

Set $p := n_0 - m_0$ and $v = \cdots v_{-2} v_{-1} | v_0 v_1 \cdots := u^{(m_0)}$. Since $u^{(n_0)} = \varphi^{n_0 - m_0}(u^{(m_0)}) = \varphi^p(v)$, inequality (40) can be rewritten as

$$\mathrm{d}\big(\varphi^p(v), v\big) < \frac{1}{2} . \tag{41}$$

The latter, together with the primitivity of the morphism φ, implies

$$\varphi^p(v_0) = v_0 w_0 \quad \text{and} \quad \varphi^p(v_{-1}) = w_{-1} v_{-1}$$

for some non-empty words $w_0, w_{-1} \in \{A, B, C\}^*$. Therefore the morphism φ^p has the fixed point

$$\lim_{n\to\infty} \varphi^{np}(v)\,.$$

Since $\varphi^{np}(v)$ is a 3iet word given by T_1, or T_2, there exist for every n a number $x^{(n)} \in [0, \alpha+\beta+\gamma]$, such that

$$\varphi^{np}(v) = u_{T_1}(x^{(n)}) \quad \text{or} \quad u_{T_2}(x^{(n)})\,.$$

Denote by x the limit of some monotonous subsequence of $(x^{(n)})_{n\in\mathbb{N}}$, i.e., $x = \lim_{n\to\infty} x^{(k_n)}$. According to Lemma 17,

$$\lim_{n\to\infty} \varphi^{np}(v) = u_{T_1}(x) \quad \text{or} \quad u_{T_2}(x)\,,$$

which means that φ^p has as its fixed point a 3iet word, namely $u_{T_1}(x)$ or $u_{T_2}(x)$, respectively. □

8 Proof of Theorem B

In the proof of Theorem B we use certain properties of discrete sets associated with 3iet words. Every 3iet word can be geometrically represented using a C&P set. On the other hand, every fixed point of a primitive morphism can be represented by a self-similar set, which is constructed using a right eigenvector of the matrix of the morphism. The crucial point in the proof of Theorem B is the fact that for a 3iet word being a fixed point of a primitive morphism these two geometric representations coincide.

We first show that the determinant of the incidence matrix of a 3iet preserving morphism is in modulus smaller than 1. For that we use the following technical lemma.

Lemma 20. *Let $\varepsilon \in (0,1)$ be a quadratic irrational number with conjugate $\varepsilon' < 0$. Let $\lambda \in (0,1)$ be a quadratic unit such that its conjugate satisfies $\lambda' > 1$ and $\lambda'\mathbb{Z}[\varepsilon'] = \mathbb{Z}[\varepsilon'] := \mathbb{Z} + \varepsilon'\mathbb{Z}$. Let us denote $\Lambda := \lambda'$, $\eta := -\varepsilon'$ and*

$$P_n(x) := \#\big(x, x + (1+2\eta)\Lambda^n\big] \cap \Sigma_{\varepsilon,\eta}(\Omega)\,,$$

where Ω is a bounded interval. Then there is a constant R such that

$$|P_n(x) - P_n(y)| \le R\,,$$

for any $x, y \in \mathbb{R}$ and $n \in \mathbb{N}$.

The proof of Lemma exploits some simple properties of C&P sets, which are however not related to infinite words. Therefore we postpone it to the appendix.

Proposition 21. *The incidence matrix $\boldsymbol{M}$ of a primitive 3iet preserving morphism φ satisfies $|\det \boldsymbol{M}| \le 1$.*

Proof. Without loss of generality we assume that φ has a 3iet fixed point u, and, moreover, that both the matrix $\boldsymbol{M}$ and its spectrum are positive. This is possible since according to Proposition 19 for any primitive 3iet preserving morphism there exists $p \in \mathbb{N}$ such that φ^p has a fixed point and $|\det \boldsymbol{M}| \le 1 \;\Leftrightarrow\; |\det \boldsymbol{M}^p| \le 1$.

Let us denote by Λ the dominant (Perron) eigenvalue of $\boldsymbol{M}$. Its second eigenvalue is by Corollary 15 equal to $\pm \det \boldsymbol{M}$, the third one is denoted by λ. A positive integer matrix cannot have 1 as its dominant eigenvalue, hence by Corollary 15, $\Lambda > 1$ is a quadratic algebraic unit such that $\Lambda' = \lambda$.

Without loss of generality we assume that a positive right eigenvector associated with the Perron eigenvalue Λ is such that the modulus of its third component is greater than the modulus of the first one. Otherwise, we use $\xi \circ \varphi \circ \xi$ instead of φ, where ξ is defined as in Example 10. Matrices corresponding to φ and $\xi\circ\varphi\circ\xi$ have the same spectrum, the first and the last component of eigenvectors being interchanged.

The fixed point u of φ is the coding of a 3iet with parameters α, β, γ, with a starting point x_0. By (5) and Corollary 6 the vector (α, β, γ) is a left eigenvector of $\boldsymbol{M}$ corresponding to Λ.

In the proof we use properties of a C&P set; we construct it in such a way that it coincides with the geometric representation of the fixed point u. Let us define parameters ε, l, c and the interval Ω by (16). Note that $(l-\varepsilon, 1-l, l-1+\varepsilon)$ is also an eigenvector to Λ, because it is just a scalar multiple of (α, β, γ), and, moreover, since Λ is a quadratic irrational number, the parameters ε, l belong to the same quadratic algebraic field $\mathbb{Q}(\Lambda)$. By x' we denote the image of $x \in \mathbb{Q}(\Lambda)$ under the Galois automorphism on $\mathbb{Q}(\Lambda)$.

Let us denote $\vec{F} = \left(\begin{smallmatrix}-\varepsilon\\ 1-2\varepsilon\\ 1-\varepsilon\end{smallmatrix}\right)$. The vector $\vec{F}$ is orthogonal to two left eigenvectors $(1,-1,1)$ and $(l-\varepsilon, 1-l, l-1+\varepsilon)$ associated with eigenvalues $\pm\det \boldsymbol{M}$ and Λ, respectively. The matrix $\boldsymbol{M}$ has three different eigenvalues, therefore $\vec{F}$ is a right eigenvector to the third eigenvalue λ.

Since the matrix $\boldsymbol{M}$ is integral, the vector $\vec{F}' := \left(\begin{smallmatrix}-\varepsilon'\\ 1-2\varepsilon'\\ 1-\varepsilon'\end{smallmatrix}\right)$ is a right eigenvector corresponding to the dominant eigenvalue $\lambda' = \Lambda$, that is,

$$\boldsymbol{M}\begin{pmatrix}-\varepsilon'\\ 1-2\varepsilon'\\ 1-\varepsilon'\end{pmatrix} = \lambda'\begin{pmatrix}-\varepsilon'\\ 1-2\varepsilon'\\ 1-\varepsilon'\end{pmatrix} = \Lambda\begin{pmatrix}-\varepsilon'\\ 1-2\varepsilon'\\ 1-\varepsilon'\end{pmatrix}. \tag{42}$$

Therefore the components of $\vec{F}'$ are either all positive or all negative. By assumption, the modulus of the third component of a right dominant eigenvector is greater than the modulus of the first one, which implies that all components of $\vec{F}'$ are positive, i.e., $-\varepsilon' > 0$.

We define a C&P set with parameters $\varepsilon, \eta, \Omega$, where ε and Ω are as above and we put $\eta := -\varepsilon'$. By Proposition 3, the distances between adjacent elements of $\Sigma_{\varepsilon,\eta}(\Omega)$ take values $\mu_A = \eta$, $\mu_B = 1 + 2\eta$, and $\mu_C = 1 + \eta$ and their ordering with respect to the origin is coded by the word u. Let $(t_n)_{n\in\mathbb{Z}}$ denote a strictly increasing sequence such that $\Sigma_{\varepsilon,\eta}(\Omega) = \{t_n \mid n \in \mathbb{Z}\}$. According to Section 2.2, this C&P set is also the geometric representation of the fixed point u of φ and

$$\#\big((\Lambda t_n, \Lambda t_{n+1}] \cap \Sigma_{\varepsilon,\eta}(\Omega)\big) = \begin{cases} |\varphi(A)| & \text{if } t_{n+1} - t_n = \mu_A = \eta\,, \\ |\varphi(B)| & \text{if } t_{n+1} - t_n = \mu_B = 1 + 2\eta\,. \\ |\varphi(B)| & \text{if } t_{n+1} - t_n = \mu_C = 1 + \eta\,. \end{cases}$$

As the fixed point of a morphism is also the fixed point of an arbitrary power of this morphism, the geometric representations of φ and φ^n are the same for any $n \in \mathbb{N}$. Since AC is a factor of any 3iet word, there exist $k, m \in \mathbb{N}$ such that

$$|\varphi^n(AC)| = \#\big((\Lambda^n t_k, \Lambda^n t_{k+2}] \cap \Sigma_{\varepsilon,\eta}(\Omega)\big)\,, \tag{43}$$

$$|\varphi^n(B)| = \#\big((\Lambda^n t_m, \Lambda^n t_{m+1}] \cap \Sigma_{\varepsilon,\eta}(\Omega)\big)\,. \tag{44}$$

By definition of the matrix of a morphism and by Corollary 16, we have $|\varphi^n(AC)| - |\varphi^n(B)| = \pm(\det \boldsymbol{M})^n$. Observe that intervals $(\Lambda^n t_k, \Lambda^n t_{k+2}]$ and $(\Lambda^n t_m, \Lambda^n t_{m+1}]$ have the same length, namely $\Lambda^n(1+2\eta)$, and that the equality $\lambda'\mathbb{Z}[\varepsilon'] = \mathbb{Z}[\varepsilon']$ holds due to (42). We can therefore use Lemma 20, which states that the difference between the right hand sides of (43) and (44) is bounded by a constant R independent of n. Putting both facts together one obtains

$$|\det \boldsymbol{M}^n| \leq R \quad \text{for any } n \in \mathbb{N}.$$

The statement follows from the fact that $\det \boldsymbol{M}$ is an integer. $\square$

Corollary 22. *The incidence matrix of a 3iet preserving morphism satisfies* $|\det \boldsymbol{M}| \leq 1$.

Proof. Consider the primitive morphism φ_0, defined in Example 11, and let us denote by $\boldsymbol{M}_0$ a power of this matrix, which is positive. Let φ be a non-primitive 3iet preserving morphism and let $\boldsymbol{M}$ be its matrix. The matrix $\boldsymbol{M}\boldsymbol{M}_0$ is positive, and thus it is the matrix of a primitive 3iet preserving morphism. By Proposition 21 we have

$$1 \geq |\det(\boldsymbol{M}\boldsymbol{M}_0)| = |\det \boldsymbol{M}|\underbrace{|\det \boldsymbol{M}_0|}_{=1} = |\det \boldsymbol{M}|\,. \qquad \square$$

Theorem B. *Let φ be a 3iet preserving morphism and let $\boldsymbol{M}$ be its incidence matrix. Then one of the following holds*

- $\det \boldsymbol{M} = 0$ *and $\varphi(u)$ is degenerated for every 3iet word u,*
- $\det \boldsymbol{M} = \pm 1$ *and $\varphi(u)$ is non-degenerated for every non-degenerated 3iet word u.*

Proof. We use notation from Lemma 12 and Remark 13. In particular, recall

$$\vec{x}_1 := \left(\begin{smallmatrix}1\\1\\0\end{smallmatrix}\right), \qquad \vec{x}_2 := \left(\begin{smallmatrix}0\\1\\1\end{smallmatrix}\right), \qquad \vec{x}_3 := \left(\begin{smallmatrix}0\\1\\0\end{smallmatrix}\right).$$

Lemma 12 states that $\boldsymbol{M}\mathcal{P} = \mathcal{P}$, where $\mathcal{P}$ is a subspace of $\mathbb{R}^3$ spanned by vectors $\vec{x}_1$, $\vec{x}_2$. Let us denote by $\mathcal{S}$ the lattice $\mathcal{S} = \mathbb{Z}\vec{x}_1 + \mathbb{Z}\vec{x}_2$. The action of the matrix $\boldsymbol{M}$ on the 2-dimensional subspace $\mathcal{P}$ has the matrix $\widetilde{M}$ from (29), which has by Lemma (14) determinant $\delta \in \{1, -1\}$. Therefore the vectors $\boldsymbol{M}\vec{x}_1$ and $\boldsymbol{M}\vec{x}_2$ form a basis of $\mathcal{S}$ as well and thus

$$\boldsymbol{M}\mathcal{S} = \mathcal{S}\,. \tag{45}$$

By an easy computation, $\boldsymbol{M}\vec{x}_3 = m_{12}\vec{x}_1 + m_{32}\vec{x}_2 + (-m_{12} - m_{32} + m_{22})\vec{x}_3$, hence by (33) we have $\boldsymbol{M}\vec{x}_3 \in \delta\Delta\vec{x}_3 + \mathcal{S}$, where $\Delta = \det \boldsymbol{M}$ as before. Moreover,

$$\left(\begin{smallmatrix}1\\1\\1\end{smallmatrix}\right) = -\vec{x}_3 + \vec{x}_1 + \vec{x}_2 \qquad \Longrightarrow \qquad \boldsymbol{M}\left(\begin{smallmatrix}1\\1\\1\end{smallmatrix}\right) \in -\delta\Delta\vec{x}_3 + \mathcal{S}\,,$$

and if we replace $\vec{x}_3$ on the right-hand side using $-\vec{x}_3 = \left(\begin{smallmatrix}1\\1\\1\end{smallmatrix}\right) - \vec{x}_1 - \vec{x}_2$ we obtain

$$\boldsymbol{M}\left(\begin{smallmatrix}1\\1\\1\end{smallmatrix}\right) \in \delta\Delta\left(\begin{smallmatrix}1\\1\\1\end{smallmatrix}\right) + \mathcal{S}\,. \tag{46}$$

<u>Case 1:</u> Let $\Delta = \det \boldsymbol{M} = 0$. Then by (46) and (45) there exist $K_1, L_1 \in \mathbb{Z}$ such that

$$\boldsymbol{M}\left(\begin{smallmatrix}1\\1\\1\end{smallmatrix}\right) = \boldsymbol{M}\left(\begin{smallmatrix}K_1\\K_1+L_1\\L_1\end{smallmatrix}\right) \quad \text{which implies} \quad (\alpha,\beta,\gamma)\boldsymbol{M}\left(\begin{smallmatrix}1\\1\\1\end{smallmatrix}\right) = (\alpha,\beta,\gamma)\boldsymbol{M}\left(\begin{smallmatrix}K_1\\K_1+L_1\\L_1\end{smallmatrix}\right),$$

for arbitrary parameters (α,β,γ). It means that $(\alpha,\beta,\gamma)\boldsymbol{M}$ are parameters of a degenerated 3iet word.

Case 2: Let $\Delta = \det \boldsymbol{M} = \pm 1$. Again, by (46) there exist $K_2, L_2 \in \mathbb{Z}$ such that

$$\boldsymbol{M}\begin{pmatrix}1\\1\\1\end{pmatrix} = \pm\begin{pmatrix}1\\1\\1\end{pmatrix} + \begin{pmatrix}K_2\\K_2+L_2\\L_2\end{pmatrix}. \tag{47}$$

We show that parameters $(\alpha,\beta,\gamma)\boldsymbol{M}$ correspond to a degenerated 3iet word only if the original parameters (α,β,γ) correspond to a degenerated 3iet word.

Let (α,β,γ) be such that $(\alpha,\beta,\gamma)\boldsymbol{M}$ are parameters of a degenerated 3iet word, i.e., there exist $K_3, L_3, H, S \in \mathbb{Z}$

$$(\alpha,\beta,\gamma)\boldsymbol{M}\begin{pmatrix}1\\1\\1\end{pmatrix} = (\alpha,\beta,\gamma)\boldsymbol{M}\begin{pmatrix}K_3\\K_3+L_3\\L_3\end{pmatrix} = (\alpha,\beta,\gamma)\begin{pmatrix}H\\H+S\\S\end{pmatrix}, \tag{48}$$

where the last equality comes from (45). Multiplying the equation (47) by (α,β,γ) from the left one obtains

$$(\alpha,\beta,\gamma)\boldsymbol{M}\begin{pmatrix}1\\1\\1\end{pmatrix} = \pm(\alpha,\beta,\gamma)\begin{pmatrix}1\\1\\1\end{pmatrix} + (\alpha,\beta,\gamma)\begin{pmatrix}K_2\\K_2+L_2\\L_2\end{pmatrix}. \tag{49}$$

Finally, comparing right-hand sides of (48) and (49) we have

$$(\alpha,\beta,\gamma)\begin{pmatrix}1\\1\\1\end{pmatrix} = \pm(\alpha,\beta,\gamma)\begin{pmatrix}K_2-H\\K_2+L_2-H-S\\L_2-S\end{pmatrix},$$

which means that (α,β,γ) are parameters of a degenerated 3iet word. ∎

9 Comments and open problems

1) We have derived that matrices of 3iet preserving morphisms belong to the monoid $\mathrm{E}(3,\mathbb{N}) := \{\boldsymbol{M} \in \mathbb{N}^{3\times 3} \mid \boldsymbol{M}\boldsymbol{E}\boldsymbol{M}^T = \pm\boldsymbol{E} \text{ and } \det \boldsymbol{M} = \pm 1\}$ where $\boldsymbol{E} = \begin{pmatrix}0&1&1\\-1&0&1\\-1&-1&0\end{pmatrix}$. Unfortunately — in contrast to the sturmian case — the opposite is not true. In fact, the monoid $\mathrm{E}(3,\mathbb{N})$ contains matrices associated with morphisms, which are not 3iet preserving. As an example one can consider the matrix $\boldsymbol{M} = \begin{pmatrix}0&2&1\\2&3&5\\3&0&5\end{pmatrix}$.

2) The mapping $\varphi \to \boldsymbol{M}_\varphi$, where φ a is 3iet preserving morphism and $\boldsymbol{M}_\varphi$ is its incidence matrix is not one-to-one. One can show that for $\begin{pmatrix}a&b\\c&d\end{pmatrix} \in \mathbb{N}^{2\times 2}$ with $ad - bc = \pm 1$ there exist $a+b+c+d-1$ different sturmian morphisms. The same question for matrices of 3iet preserving morphisms is not solved.

3) Unlike the free monoid $\mathrm{SL}(2,\mathbb{N}) = \{\boldsymbol{M} \in \mathbb{N}^{2\times 2} \mid \det \boldsymbol{M} = 1\}$, which is generated by two matrices $\begin{pmatrix}1&1\\0&1\end{pmatrix}$ and $\begin{pmatrix}1&0\\1&1\end{pmatrix}$, the monoid $\mathrm{SL}(3,\mathbb{N})$ is not free, and, moreover, it is not finitely generated [19, Appendix A]. It would be interesting to derive similar results for the monoid $\mathrm{E}(3,\mathbb{N})$.

4) Even though the aim of this paper is not to investigate explicite prescriptions of 3iet preserving morphisms, we can still provide some information about it, based on our results. It follows from the proof of Lemma 14 that for every 3iet preserving morphism $\varphi : \{A,B,C\}^* \to \{A,B,C\}^*$ the morphism given by $A \mapsto \sigma_{A,B} \circ \varphi(A)$ and $B \mapsto \sigma_{A,B} \circ \varphi(B)$ is sturmian, where $\sigma_{A,B}$: $A \mapsto A$, $B \mapsto AB$, $C \mapsto B$; analogously for morphisms $\sigma_{A,C}$ and $\sigma_{B,C}$.

5) In this paper we were not at all interested in the characterization of 3iet words, which are fixed points of primitive morphisms, that is, of 3iet words u such that there exists a primitive morphisms φ for which $\varphi(u) = u$. This question is completely solved for sturmian words [24, 3]. Adamczewski [1] studied for 3iet words a weaker property, the so-called primitive substitutivity. An infinite word u over an alphabet $\mathcal{A}$ is said to be primitively substitutive if there exists a word v over an alphabet $\mathcal{B}$, which is a fixed point of a primitive morphism, and a morphism $\psi : \mathcal{B}^* \to \mathcal{A}^*$ such that $\psi(v) = u$. Adamczewski, using results of Boshernitzan and Carroll [7], proved that a non-degenerated 3iet word is primitively substitutive if and only if normalized parameters ε, l, c (see (16)) of the corresponding transformation belong to the same quadratic field. Similar study can be foud in [12].

Acknowledgements

The authors acknowledge financial support by Czech Science Foundation GA ČR 201/05/0169, and by the grant LC06002 of the Ministry of Education, Youth, and Sports of the Czech Republic.

A Proof of Lemma 20

In this Appendix we prove Lemma 20, which is rather technical. The proof uses the four following claims.

Claim 1. Let ε, η be irrational numbers, $\varepsilon \neq -\eta$, and let Ω_1, Ω_2 be arbitrary bounded intervals. Then $\#\big(\Omega_1 \cap \Sigma_{\varepsilon,\eta}(\Omega_2)\big) = \#\big(\Omega_2 \cap \Sigma_{\eta,\varepsilon}(\Omega_1)\big)$.

Proof.

$$\begin{aligned}\#\big(\Omega_1 \cap \Sigma_{\varepsilon,\eta}(\Omega_2)\big) &= \#\{a + b\eta \mid a, b \in \mathbb{Z},\ a + b\eta \in \Omega_1,\ a - b\varepsilon \in \Omega_2\} = \\ &= \#\{a + c\varepsilon \mid a, c \in \mathbb{Z},\ a + c\varepsilon \in \Omega_2,\ a - c\eta \in \Omega_1\} = \\ &= \#\big(\Omega_2 \cap \Sigma_{\eta,\varepsilon}(\Omega_1)\big)\,. \end{aligned}$$

□

Claim 2. Let ε be a quadratic irrational number with conjugate ε'. Let λ be a quadratic unit whose conjugate λ' satisfies $\lambda'\mathbb{Z}[\varepsilon'] = \mathbb{Z}[\varepsilon']$. Then $\lambda'\Sigma_{\varepsilon,-\varepsilon'}(\Omega) = \Sigma_{\varepsilon,-\varepsilon'}(\lambda\Omega)$.

Proof. Let us consider $x = a - b\varepsilon \in \mathbb{Z}[\varepsilon]$. If we denote $\eta = -\varepsilon'$, the number $a + b\eta = a - b\varepsilon' \in \mathbb{Z}[\varepsilon']$ is the image of x under the Galois automorphism and therefore we denote it by $x' = a - b\varepsilon'$.

Note that the condition $\lambda'\mathbb{Z}[\varepsilon'] = \mathbb{Z}[\varepsilon']$ is equivalent to the condition $\lambda\mathbb{Z}[\varepsilon] = \mathbb{Z}[\varepsilon]$, and that these two equalities imply that the mappings $x' \mapsto \lambda' x'$ and $x \mapsto \lambda x$ are bijections on $\mathbb{Z}[\varepsilon']$ and $\mathbb{Z}[\varepsilon]$, respectively.

By definition of a C&P set we have

$$\Sigma_{\varepsilon,-\varepsilon'}(\Omega) = \{x' \in \mathbb{Z}[\varepsilon'] \mid x \in \Omega\}\,.$$

We derive

$$\begin{aligned}\lambda'\Sigma_{\varepsilon,-\varepsilon'}(\Omega) &= \lambda'\{x' \in \mathbb{Z}[\varepsilon'] \mid x \in \Omega\} = \{\lambda' x' \in \mathbb{Z}[\varepsilon'] \mid \lambda x \in \lambda\Omega\} = \\ &= \{y' \in \mathbb{Z}[\varepsilon'] \mid y \in \lambda\Omega\} = \Sigma_{\varepsilon,-\varepsilon'}(\lambda\Omega)\,.\end{aligned}$$

□

The following Claim is given without the proof, since it is just a special case of Proposition 6.2 in [11].

Claim 3. Let $\hat\varepsilon,\hat\eta$ be irrational numbers, $\hat\varepsilon\neq-\hat\eta$, and let $\hat\Omega$ be an arbitrary bounded interval. Then

$$\Sigma_{\hat\varepsilon,\hat\eta}((1+2\hat\varepsilon)\hat\Omega)=(1-2\hat\eta)\Sigma_{\frac{\hat\varepsilon}{1-2\hat\varepsilon},\frac{\hat\eta}{1+2\hat\eta}}(\hat\Omega)\,.$$

Claim 4. Let $\tilde\varepsilon,\tilde\eta$ be irrational numbers such that $\tilde\varepsilon\neq-\tilde\eta$. Let $z\in\mathbb{R}$ and let J be a bounded interval. We denote $Q(J,z):=\#(J\cap\Sigma_{\tilde\varepsilon,\tilde\eta}(z-1,z])$. Then there is a constant R such that $|Q(J,z)-Q(J,t)|\leq R$ for every $z,t\in\mathbb{R}$ and for every interval J.

Proof. The condition $a-b\tilde\varepsilon\in(z-1,z]$, where $a,b\in\mathbb{Z}$, can be equivalently rewritten as $a=\lfloor z+b\tilde\varepsilon\rfloor=z+b\tilde\varepsilon-\{z+b\tilde\varepsilon\}$. Hence

$$\Sigma_{\tilde\varepsilon,\tilde\eta}(z-1,z]=\{b(\tilde\varepsilon+\tilde\eta)+z-\{z+b\tilde\varepsilon\}\mid b\in\mathbb{Z}\}\,.$$

We consider the interval J with boundary points $c,c+l$, where $c,l\in\mathbb{R}$ and $l>0$. If the point $b(\tilde\varepsilon+\tilde\eta)+z-\{z+b\tilde\varepsilon\}$ belongs to the set $J\cap\Sigma_{\tilde\varepsilon,\tilde\eta}(z-1,z]$, then $c-z\leq b(\tilde\varepsilon+\tilde\eta)\leq c+l-z+1$. On the other hand, if $c-z+1<b(\tilde\varepsilon+\tilde\eta)<c+l-z$ then the point $b(\tilde\varepsilon+\tilde\eta)+z-\{z+b\tilde\varepsilon\}$ belongs to $J\cap\Sigma_{\tilde\varepsilon,\tilde\eta}(z-1,z]$. It means that the number of points in the set $J\cap\Sigma_{\tilde\varepsilon,\tilde\eta}(z-1,z]$ is at least $\left\lfloor\frac{l-1}{\tilde\varepsilon+\tilde\eta}\right\rfloor$ and at most $\left\lceil\frac{l+1}{\tilde\varepsilon+\tilde\eta}\right\rceil$, and hence

$$\left\lfloor\frac{l-1}{\tilde\varepsilon+\tilde\eta}\right\rfloor\leq Q(J,z)\leq\left\lceil\frac{l+1}{\tilde\varepsilon+\tilde\eta}\right\rceil.$$

Note that the bounds on $Q(J,z)$ do not depend on z, and thus the same estimate holds for $Q(J,t)$. Therefore

$$|Q(J,z)-Q(J,t)|\leq\left\lceil\frac{l+1}{\tilde\varepsilon+\tilde\eta}\right\rceil-\left\lfloor\frac{l-1}{\tilde\varepsilon+\tilde\eta}\right\rfloor\leq2\Big(1+\frac{1}{\tilde\varepsilon+\tilde\eta}\Big)=:R\,.$$

□

Now we are in the position to conclude the proof of Lemma 20.

Proof of Lemma 20. Let us recall the definition of $P_n(x)=\#\Big(\big(x,x+(1+2\eta)\Lambda^n\big]\cap\Sigma_{\varepsilon,\eta}(\Omega)\Big)$. By Claim 1, we have

$$P_n(x)=\#\Big(\Omega\cap\Sigma_{\eta,\varepsilon}(x,x+(1+2\eta)\Lambda^n]\Big)=\#\Big(\Lambda^n\Omega\cap\Lambda^n\Sigma_{\eta,\varepsilon}(x,x+(1+2\eta)\Lambda^n]\Big)\,.$$

As $\eta=-\varepsilon'$ and $\lambda'=\Lambda$, we have, by Claim 2,

$$P_n(x)=\#\Big(\Lambda^n\Omega\cap\Sigma_{\eta,\varepsilon}\big(\lambda^nx,\lambda^nx+(1+2\eta)\big]\Big)\,,$$

where we used $\lambda\lambda'=\lambda\Lambda=1$. Claim 3 further implies

$$P_n(x)=\#\bigg(\Lambda^n\Omega\cap(1-2\varepsilon)\Sigma_{\frac{\eta}{1-2\eta},\frac{\varepsilon}{1+2\varepsilon}}\Big(\frac{\lambda^nx}{1+2\eta},\frac{\lambda^nx}{1+2\eta}+1\Big]\bigg)\,.$$

Thus $P_n(x)=Q\big(J,\frac{\lambda^nx}{1+2\eta}+1\big)$ as defined in Claim 4, where $J=\frac{1}{1-2\varepsilon}\Lambda^n\Omega$, $\tilde\varepsilon=\frac{\eta}{1-2\eta}$ and $\tilde\eta=\frac{\varepsilon}{1+2\varepsilon}$. The statement of Lemma follows by application of Claim 4. □

References

[1] B. Adamczewski, Codages de rotations et phénomènes d'autosimilarité, J. Théor. Nombres Bordeaux 14 (2) (2002) 351–386.

[2] P. Arnoux, G. Rauzy, Représentation géométrique de suites de complexité $2n+1$, Bull. Soc. Math. France 119 (2) (1991) 199–215.

[3] P. Baláži, Z. Masáková, E. Pelantová, Complete characterization of substitution invariant Sturmian sequences, Integers 5 (1) (2005) A14, 23 pp. (electronic).

[4] P. Baláži, Z. Masáková, E. Pelantová, Factor versus palindromic complexity of uniformly recurrent infinite words, to apper in Theor. Comp. Sci. (2006).

[5] J. Berstel, Recent results on extensions of Sturmian words, Internat. J. Algebra Comput. 12 (1-2) (2002) 371–385, International Conference on Geometric and Combinatorial Methods in Group Theory and Semigroup Theory (Lincoln, NE, 2000).

[6] J. Berstel, P. Séébold, Morphismes de Sturm, Bull. Belg. Math. Soc. Simon Stevin 1 (2) (1994) 175–189, Journées Montoises (Mons, 1992).

[7] M. D. Boshernitzan, C. R. Carroll, An extension of Lagrange's theorem to interval exchange transformations over quadratic fields, J. Anal. Math. 72 (1997) 21–44.

[8] J. Cassaigne, Sequences with grouped factors, in: S. Bozapalidis (ed.), Proceedings of the Third International Conference Developments in Language Theory (Thessaloniki 1997), Aristotle University of Thessaloniki, 1997.

[9] D. Damanik, L. Q. Zamboni, Combinatorial properties of Arnoux-Rauzy subshifts and applications to Schrödinger operators, Rev. Math. Phys. 15 (7) (2003) 745–763.

[10] S. Ferenczi, C. Holton, L. Q. Zamboni, Structure of three-interval exchange transformations. II. A combinatorial description of the trajectories, J. Anal. Math. 89 (2003) 239–276.

[11] L.-S. Guimond, Z. Masáková, E. Pelantová, Combinatorial properties of infinite words associated with cut-and-project sequences, J. Théor. Nombres Bordeaux 15 (3) (2003) 697–725.

[12] E. O. Harriss, J. S. Lamb, One-dimensional substitution tilings with an interval projection structure, preprint DynamIC 2006-01, Arxiv math. DS/0601187 (2006).

[13] N. Jacobson, Lie algebras, Interscience Tracts in Pure and Applied Mathematics, No. 10, Interscience Publishers (a division of John Wiley & Sons), New York-London, 1962.

[14] A. B. Katok, A. M. Stepin, Approximations in ergodic theory, Uspehi Mat. Nauk 22 (5 (137)) (1967) 81–106.

[15] M. Keane, Interval exchange transformations, Math. Z. 141 (1975) 25–31.

[16] M. Lothaire, Algebraic Combinatorics on Words, vol. 90 of Encyclopedia of Mathematics and its Applications, Cambridge University Press, Cambridge, 2002.

[17] F. Mignosi, P. Séébold, Morphismes sturmiens et règles de Rauzy, J. Théor. Nombres Bordeaux 5 (2) (1993) 221–233.

[18] M. Morse, G. A. Hedlund, Symbolic dynamics II. Sturmian trajectories, Amer. J. Math. 62 (1940) 1–42.

[19] N. Pytheas Fogg, Substitutions in Dynamics, Arithmetics and Combinatorics, vol. 1794 of Lecture Notes in Mathematics, Springer Verlag, 2002.

[20] G. Rauzy, Échanges d'intervalles et transformations induites, Acta Arith. 34 (4) (1979) 315–328.

[21] P. Séébold, Fibonacci morphisms and Sturmian words, Theoret. Comput. Sci. 88 (2) (1991) 365–384.

[22] W. A. Veech, The metric theory of interval exchange transformations I, II, III, Amer. J. Math. 106 (6) (1984) 1331–1422.

[23] A. M. Vershik, A. N. Livshits, Adic models of ergodic transformations, spectral theory, substitutions, and related topics, in: Representation theory and dynamical systems, vol. 9 of Adv. Soviet Math., Amer. Math. Soc., Providence, RI, 1992, pp. 185–204.

[24] S.-I. Yasutomi, On Sturmian sequences which are invariant under some substitutions, in: Number theory and its applications (Kyoto, 1997), vol. 2 of Dev. Math., Kluwer Acad. Publ., Dordrecht, 1999, pp. 347–373.